Math 241 practice problems

Practice problems, Week of September 23

1. (a) Show that for x in the interval $[0,1]$ and $t \geq 0$, the function

$$u(x,t) = (t+1)^{-\frac{1}{2}} e^{-\frac{x^2}{4(t+1)}}$$

is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

(b) Give one example of a homogeneous boundary condition that this solution satisfies on the specified domain.

2. Consider the following PDE posed for an "unknown" function $u(x,t)$ defined for all $x \in [0,L]$ and all $t \geq 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} + u \quad \text{with boundary conditions} \quad u(0,t) = 0 = u(L,t) + \frac{\partial u}{\partial x}(L,t).$$

Apply the method of separation of variables to determine what differential equations are implied for functions of x and t and what boundary conditions (if any) are necessary for each of those ODEs. You do not need to solve these ODEs.

3. (a) Compute the Fourier cosine series for the function $f(x) = x^2$ on the interval $[0,\pi]$. Fully simplify your answer (i.e., the formula for the coefficients should not involve sines or cosines).

(b) Does the Fourier cosine series converge to the function f at the point $x = 0$? Justify your answer.

(c) Sketch the values of the Fourier cosine series of f on the interval $[-\pi, 2\pi]$, marking any points of discontinuity.

(d) Demonstrate from your calculations that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

4. Let $u(x,t)$ be the temperature in a rod, $0 < x < 2$ the satisfies the initial and boundary value problem

$$\begin{aligned} &u_t = u_{xx} + cx && \text{for} \quad 0 < x < 2, \quad t > 0 \\ &u_x(0,t) = 0 && \text{and} \quad u_x(2,t) = -2, \\ &u(x,0) = \frac{8}{\pi}\cos\frac{\pi}{4}x, && \end{aligned}$$

where c is a constant. Denote the thermal energy in the rod by $H(t) = \int_0^2 u(x,t)\,dx$;

(a) What is the physical meaning of the boundary condition $u_x(0,t) = 0$?

(b) Compute $\dfrac{dH}{dt}$ (it will involve the constant c).

(c) Use this to compute $H(t)$.

(d) For which value(s) of c does the limit $v(x) \lim_{t\to\infty} u(x,t)$ exist, and what is this limit?

5. Find a series solution of a function $u(x,y)$ defined on the annulus $\{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 4\}$ which satisfies the Laplace equation

$$\nabla^2 u(x,y) = \triangle u(x,y) = 0, \quad u(x,y) = \begin{cases} 0 & \text{if } x^2+y^2 = 1 \\ 1 & \text{if } x^2+y^2 = 4,\ y > 0 \\ 0 & \text{if } x^2+y^2 = 4,\ y < 0. \end{cases}$$

Fully simplify your answer.

6. Find a series solution to the following problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ for } x \in [0,\pi] \text{ and } t \geq 0,$$

$$\text{BCs: } u(0,t) = e^{-\frac{t}{4}}, u(\pi,t) = 0, \qquad \text{ICs: } u(x,0) = 0.$$

Simplify all constants to the point that they no longer involve trigonometric functions. (Hint: $v(x,t) = e^{-\frac{t}{4}} \cos\frac{x}{2}$ solves the PDE and the boundary conditions, but not the initial conditions.)

7. Find the harmonic function $u(x,y)$ (so $\nabla^2 u = 0$) on the unit disk $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ that satisfies the boundary condition

$$u(\cos\theta, \sin\theta) = 2 + 3\sin 4\theta \quad \forall\, \theta \in \mathbb{R}.$$

8. Suppose that $u(x,y)$ is a harmonic function on the unit disk $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ such that

$$u(\cos\theta, \sin\theta) = \cos^4\theta + \sin^4\theta \quad \forall\, \theta \in \mathbb{R}.$$

(a) Compute $u(0,0)$.

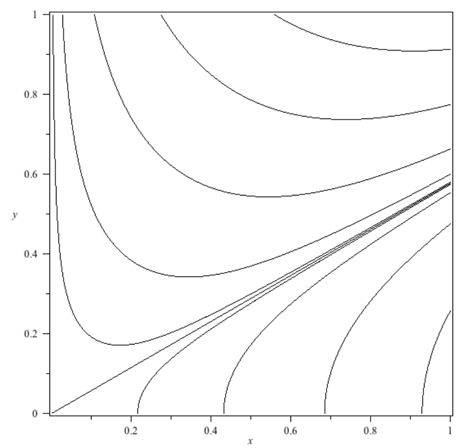
(b) Determine

$$\max\nolimits_{x^2+y^2\leq 1} u(x,y)$$

and

$$\min\nolimits_{x^2+y^2\leq 1} u(x,y).$$

9. Below you will find a contour plot of a harmonic function on the unit square (i.e., the harmonic function takes a different constant value on each of the curves). On one side of the square, the harmonic function (when considered an equilibrium solution of the heat equation) has an insulated boundary condition. On another side, there is a constant temperature boundary condition. On the two remaining sides, neither a constant temperature nor a constant flux condition is imposed. Which sides are which? Explain your reasoning.



10. Let $f(x)$ be a piecewise smooth periodic function on $\mathbb{R}$ with period 1 such that $f(x)$ is smooth on the open interval $(0,1)$. Suppose that $f(x)$ satisfies the differential equation

$$\left(\tfrac{d^2}{dx^2} + \pi^2\right) f(x) = 0,$$

and

$$\lim_{x\to 0+} f(x) = \lim_{x\to 1-} f(x) + 1, \qquad \lim_{x\to 0+} f'(x) = \lim_{x\to 1-} f'(x).$$

Determine the Fourier series of the function

$$g(x) := f(x) - \int_0^1 f(x)\,dx.$$