5. General harmonic functions $u(x,y)$ on the annulus $\{1 \le x^2+y^2 \le 4\}$ in polar coordinates, in the form of an infinite series (which you get by separation of variables is:

$$u(r\cos\theta, r\sin\theta) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n r^n e^{\sqrt{-1}n\theta} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} b_n r^{-n} e^{\sqrt{-1}n\theta} + a_0 + b_0 \log r$$

The Fourier series expansion of the function

$$f(\theta) = \begin{cases} 1 & 0<\theta<\pi \\ 0 & \pi<\theta<2\pi \end{cases}$$

is $\sum_{n\in\mathbb{Z}} c_n e^{\sqrt{-1}n\theta}$

$$c_n = \frac{1}{2\pi}\int_0^\pi 1\cdot e^{-\sqrt{-1}n\theta} = \frac{1}{2\pi}\cdot\frac{\sqrt{-1}}{n}\, e^{-\sqrt{-1}n\theta}\Big]_0^\pi$$
$n\neq 0$
$$= \frac{\sqrt{-1}}{2\pi n}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{if } n \text{ even} \\ \dfrac{1}{\pi n\sqrt{-1}} & \text{if } n \text{ odd}\end{cases}$$

$$c_0 = \frac{1}{2}$$

i.e. $$f(\theta) = \frac{1}{2} + \sum_{n\text{ odd}} \frac{1}{\pi n\sqrt{-1}} e^{\pi\sqrt{-1}n}$$

The boundary condition for $u(r\cos\theta, r\sin\theta)$ becomes:

$r=1 \rightsquigarrow \quad a_n + b_n = 0 \quad \forall n\neq 0 \quad b_n = -a_n$

$r=2 \rightsquigarrow \quad 2^n a_n + 2^{-n} b_n = \frac{1}{\pi n\sqrt{-1}} \quad \forall n\neq 0 \quad (2^n - 2^{-n})\, a_n = \frac{1}{\pi n\sqrt{-1}} \quad \forall n\neq 0$

For a_0 and b_0: $\quad r=1 \rightsquigarrow a_0 = 0$

$\qquad r=2 \rightsquigarrow b_0 \log_e 2 = \frac{1}{2} \rightsquigarrow b_0 = \frac{1}{2\log 2}$

Conclusion:

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\log 2} + \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{\pi\sqrt{-1}\,(2^n-2^{-n})\cdot n}\, e^{\sqrt{-1}n\theta} + \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{-1}{\pi\sqrt{-1}\,(2^n-2^{-n})\, n}\, e^{-\sqrt{-1}n\theta}$$

Note on separation of variable:

A product solution $f(r)\, g(t)$ of $\Delta(f(r)\cdot g(t)) = 0$ need to satisfy

$$\frac{\left(r\frac{d}{dr}\right)^2 f(r)}{f(r)} = n^2 = -\frac{\frac{d^2}{d\theta^2} g(\theta)}{g(\theta)}, \quad n\in\mathbb{Z}.$$

For $n=0$, the ODE for $f(r)$ has two indep. sol^n: 1 and $\log r$