Math 241 practice problems

Practice problems, Week of October 28

1. Suppose that $u(x,y,t)$ is the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

on the unit square (i.e., $0 \leq x \leq 1$ and $0 \leq y \leq 1$) subject to free boundary conditions on the top and bottom and homogeneous fixed boundary conditions on the left and right sides. Assuming that u satisfies the initial conditions

$$u(x,y,0) = 0 \text{ and } \frac{\partial u}{\partial t}(x,y,0) = 1,$$

find a series expansion for $u(x,y,t)$ when $t \geq 0$. Is the solution periodic in time?

(Recall that "free boundary condition" means that the derivative in the normal direction vanishes along the part of the boundary in question, while "homogeneous fixed boundary condition" means that the value of the function is zero on the part of the boundary in question.)

2. Identify all solutions of the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u$$

on the half disk $x^2 + y^2 \leq 1$ & $y \geq 0$ with homogeneous Dirichlet boundary condition.

(Recall that "homogeneous Dirichlet boundary condition" means that the value of the unknown function $u(x,y)$ at the boundary is identically zero.)

3. Give a series solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

on the half cylinder $x^2 + y^2 \leq 1$, $y \geq 0$, and $0 \leq z \leq 1$ assuming that the temperature is zero on the lateral sides and the bottom but arbitrary on the top (the top being the face where $z = 1$). Simplify your answer as much as possible; you do not need to describe exactly how the coefficients depend on the temperature function on the top.

4. Suppose that $u(x,y,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

for $t \geq 0$ and (x, y) in the circular disk $D = \{x^2 + y^2 \leq 1\}$, and $u(x, y)$ satisfies the boundary condition

$$u(x, y, t) = 0 \quad \text{when} \quad x^2 + y^2 = 1.$$

Suppose moreover that the function $u(x, y, t)$ is symmetric about the center $(0, 0)$ of the disk; i.e. $u(x, y, t) = u(x_1, y_1, t)$ whenever $x^2 + y^2 = x_1^2 + y_1^2$. Suppose that $u(x, y, 0) = f(\sqrt{x^2 + y^2})$, where $f(r)$ is a smooth function on $[0, 1]$.

(a) Write down a series solution of $u(x, y, t)$ in terms of the Fourier–Bessel expansion coefficients of $f(r)$.

(b) (extra credit) Let $c > 0$ be a positive number. What is the condition (on the initial value $f(r)$) for

$$\lim_{t \to \infty} e^{ct} u(x, y, t) = 0$$

for all $(x, y) \in D$?

5. It is a fact that for any two positive integers n and m are positive integers with $0 < m < n$ (note equality is not allowed), the function

$$\phi_{nm}(x, y) := \sin n\pi x \sin m\pi y - (-1)^{n+m} \sin m\pi x \sin n\pi y$$

satisfies the Helmholtz equation $\nabla^2 \phi = -\lambda \phi$ with $\lambda = \pi^2(n^2 + m^2)$ on the triangle defined by $x \geq 0, y \geq 0$, and $x + y \leq 1$, subject to $\phi = 0$ on the boundary of the triangle.

(a) Show that the eigenfunctions ϕ_{nm} are mutually orthogonal.

(b) It is known that the family of functions ϕ_{nm} is *complete*, in the sense that every continuous and piecewise smooth function on the triangle can be written as a convergent infinite series of the form $\sum a_{nm} \phi_{nm}(x)$.

Use the above fact to write the general series solution to the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + 5\pi^2 u$$

on the same triangle with the same boundary conditions. and give a formula for computing the coefficients in terms of the initial data.

(c) What is the behavior of solutions in (b) as $t \to \infty$?

6. Find the solution (in series form) to the following heat equation inside the three-dimensional box $[0, L] \times [0, W] \times [0, H]$:

$$\frac{\partial u}{\partial t} = k \nabla^2 u.$$

Assume the insulated boundary condition on all sides.

(a) Find the general solution of this equation in the form of an infinite series.

(b) Find the coefficients for the solution whose initial condition is

$$u(x,y,z,0) = 357\cos\frac{\pi x}{L}\cos\frac{\pi y}{W}\cos\frac{\pi z}{H} - 111\cos\frac{3\pi x}{L}\cos\frac{5\pi y}{W}\cos\frac{7\pi z}{H}.$$

7. (a) Express the following boundary value problem

$$\frac{d^2\phi}{dx^2} - 4x\frac{d\phi}{dx} = (-8x^2 - 2 - \lambda)\phi, \quad x \in [0, L], \quad \phi(0) = 0 = \phi(L)$$

in the standard Sturm–Liouville form. (Hint: multiply the above equation by a suitable function $g(x)$.)

(b) For which values of L is the above a regular Sturm–Liouville problem? Write the orthogonality relation satisfied by the eigenfunctions.

(c) Are there negative eigenvalues? Fully justify your answer.

8. Suppose that $u(x,y,t)$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

on the unit square $\{(x,y) : 0 \le x, y \le 1\}$, $t \ge 0$, and satisfies the homogeneous Neumann boundary condition (i.e. all derivatives in the normal direction vanish along the boundary).

(a) Find the general solution of written as an infinite series.

(b) Find the solution which satisfies the initial conditions

$$u(x,y,0) = \cos(3\pi x)\,\cos(4\pi y), \qquad \frac{\partial u}{\partial t}(x,y,0) = 1.$$

9. Consider the boundary value problem

$$\phi'' + (2-4x)\phi' + \lambda\phi = 0, \quad \phi(0) = \phi(1) = 0.$$

(a) Rewrite the above equation in Sturm-Liouville form;

(b) Verify that $\phi(x) = x(1-x)$ is an eigenfunction for this problem, and compute its eigenvalue λ;

(c) (extra credit) Prove that the eigenvalue λ obtained in b) is the first eigenvalue λ_1 of this Sturm-Liouville problem.

10. (extra credit. Part (a) is preparatory for parts (b) and (c). Parts (b) and (c) are related but logically independent.)

(a) Show that

$$\int x^3 J_0(x)\,dx = x(x^2-4)J_1(x) + 2x^2 J_0(x) + C$$

(Hint: We have shown in class a series of recurrence relations between Bessel functions of integer order. Two among them are

$$\int J_1 X)\,dx = -J_0(x) + C, \qquad \int x\,J_0(x)\,dx = x\,J_1(x) + C.$$

These relations, plus integration by part, give closed forms of indefinite integrals $\int x^{2k-1} J_0(x)\,dx$ in terms of $J_0(x)$ and $J_1(x)$. In contrast there is no such formula for $\int J_0(x)\,dx$.)

(b) The function x^2 on $[0,1]$ has a Fourier–Bessel expansion

$$x^2 = \sum_{i=1}^{\infty} A_i\,J_0(\alpha_i x),$$

where

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$$

are the positive zeros of $J_0(x)$. Recall that

$$\int_0^1 x\,J_0(ax)^2\,dx = \tfrac{1}{2}\big(J_0^2(ax) + J_1^2(ax)\big) \quad \forall\, a \in \mathbb{R}.$$

Find a closed-form expression of

$$\sum_{i=1}^{\infty} A_i^2 J_1(\alpha_i)^2$$

(c) Show that

$$x^2 = 2\sum_{i=1}^{\infty} \frac{\alpha_i^2 - 4}{\alpha_i^3 J_1(\alpha_i)}\,J_0(\alpha_i x).$$

What does (b) say, given the explicit formula for the A_i's?

(d) Can you find an explicit formula for the indefinite integral

$$\int x\,\log x\,dx?$$

If you succeed, then you can find an explicit expression for the Fourier-Bessel expansion of the function $\log(1/x)$ on $[0,1]$.