Math 241 practice problems

Practice problems, Week of November 25

1. Find a solution of the Poisson equation

$$\triangle u = \sqrt{x^2+y^2}, \quad u(2\cos\theta, 2\sin\theta) = 16\cos(3\theta) \;\; \forall\, \theta \in \mathbb{R}$$

where the unknown function $u(x,y)$ is defined on the circular disk

$$\{(x,y) \in \mathbb{R} \mid x^2+y^2 \leq 4\}.$$

2. Find a function $u(x,y,z)$ defined on the circular cylinder

$$D := \{(x,y,z) : x^2+y^2 \leq a^2, \; 0 \leq z \leq h\},$$

where $a > 0$ is a constant, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

u is bounded,

$$u(x,y,z) = \cos(z) - 3\cos(3z) + 5\cos(5z) \;\text{ if }\; x^2+y^2 = a^2,$$

and

$$\frac{\partial u}{\partial z}(x,y,0) = \frac{\partial u}{\partial z}(x,y,h) = 0$$

3. Consider the one-dimensional wave equation with time dependent "forcing"

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \cos(\omega t), \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(1,t)$$

where the unknown function $u(x,t)$ is defined for $0 \leq x \leq 1, \; t \geq 0$.

(a) For what values of ω does resonance occur?

(b) Give an example of a resonant solution.

(c) Find the solution when there is no resonance.

4. Find a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x^2}, \quad 0 \leq x \leq 1, \; t \geq 1$$

such that

$$u(x,0) = \cos(7\pi x/2) \; \forall\, x \in [0,1], \;\; \frac{\partial u}{\partial x}(0,t) = 0, \; u(1,t) = 0 \; \forall\, t \geq 0.$$

5. Find a solution $u(x, y)$ of the Laplace equation defined on the upper half-plane such that

$$\frac{\partial u}{\partial y}(x, 0) = \frac{2x}{(1 + x^2)^2}.$$

[Hint: $\frac{d}{dx}\frac{1}{1+x^2} = -\frac{2x}{(1+x^2)^2}$.]

6. Find a solution $u(x, t)$ of the wave equation

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = \frac{\partial u}{\partial x^2}$$

defined on $\{(x, t) : x \in \mathbb{R},\ t \geq 0\}$ which satisfies the initial conditions

$$u(x, 0) = f(x),\ \frac{\partial u}{\partial t}(x, 0) = 0,$$

where $f(x)$ is a given function on $\mathbb{R}$.

7. Let $u(x, y, z)$ be a function on the box $[0, 1] \times [0, 1] \times [0, 1]$ which satisfies the Laplace equation $\triangle u = 0$, and equals $\sin(3\pi x)\sin(5\pi y)$ on the face $z = 0$, and is equal to 0 on all remaining faces. What is $u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$?

8. Let u be a function on the unit sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$, and let $u(\phi, \theta)$ be this function expressed in spherical coordinates. Let

$$u(\phi, \theta) = \sum_{n=0}^{\infty} \sum_{-n \leq k \leq n} a_{n,k}\, P_n^{|k|}(\cos\phi) e^{\sqrt{-1}k\theta}$$

be the expansion of u as a sum of spherical harmonics $P_n^{|k|}(\cos\phi)e^{\sqrt{-1}k\theta}$, where P_n^m are the Legendre functions. (Note that m is a parameter for the Legendre function, so that $P_n^m(x)$ is not the m-th power of the Legendre polynomial $P_n(x)$.) Find an explicit expression of the surface integral

$$\int\int_S u\, dA$$

of u in terms of the expansion coefficients $a_{n,k}$.

9. Find a solution of the Poisson equation

$$\triangle u = x - 1$$

on the cube $[0, 1] \times [0, 1] \times [0, 1]$ such that u vanishes on the boundary of the cube.

10. (a) Find a function u defined on the upper half-sphere

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}$$

such that

$$\triangle u = 0, \qquad u(x,y,0) = 0 \ \text{ if } x^2+y^2 \le 1$$

and

$$u(\sin\phi\,\cos\theta, \sin\phi\,\sin\theta, \cos\phi) = 2\cos\phi \ \text{ if } \ 0 \le \phi \le \tfrac{\pi}{2}, \theta \in \mathbb{R}.$$

(b) (extra credit) Find a function v defined on the upper hemisphere $\{(x,y,z) : x^2 + y^2 + z^2 \le 1,\ z \ge 0\}$ such that

$$\triangle v = 0, \qquad v(x,y,0) = 0 \ \text{ if } x^2+y^2 \le 1$$

and

$$v(\sin\phi\,\cos\theta, \sin\phi\,\sin\theta, \cos\phi) = 1 \ \text{ if } \ 0 \le \phi < \tfrac{\pi}{2}, \theta \in \mathbb{R}.$$

Note that the value of the unknown function v at the boundary of the solid half-sphere is discontinuous. Accordingly the meaning of the boundary conditions is to be interpreted somewhat loosely.

11. (a) Find a function $u(x,t)$ defined for all $x \in \mathbb{R}$ and all $t \ge 0$ such that

$$\frac{\partial u}{\partial t} = 2\frac{\partial u}{\partial x} - u, \quad u(x,0) = 7e^{-x^2}.$$

[Hint: Let $U(\omega,t)$ be the Fourier transform of $u(x,t)$ in the variable x.]

(b) (extra credit) Try to use the Fourier transform method to find a function $v(x,t)$ defined for all $x \in \mathbb{R}$ and all $t \ge 0$ such that

$$\frac{\partial v}{\partial t} = 2\frac{\partial v}{\partial x} - v, \quad v(x,0) = 7x.$$

If you try to carry out a similar computation as in part (a), you will need to either find the Fourier transform of the function $7x$, or the inverse Fourier transform of $e^{a\sqrt{-1}\omega}$ for some constant a, or both. Either will involve the Dirac δ-function. Note that the usual integral for the Fourier transform of $7x$ is $\frac{1}{2\pi}\int_{-\infty}^{\infty} 7xe^{\sqrt{-1}x\omega}\,dx$, which is *divergent* for every $\omega \in \mathbb{R}$.

12. Find a solution of the boundary value problem for the equation

$$\triangle v + v = 0,$$

in a solid sphere of radius 1, such that the restriction of v to the boundary sphere in spherical coordinates is equal to

$$v|_{r=1}(\phi,\theta) = \sin^2(\phi)\cos(3\theta).$$