Solution to Problem 12 of Practice Problems

12 v: defined on the unit ball, expressed in spherical coord

$$\Delta v + v = 0$$

$$v\big|_{S^2}(\varphi, \theta) = \sin^2(\phi) \cdot \cos(3\theta)$$

Note: The actual solution is messy, i.e. without a "clean" final answer

Step 1. Reduce to a problem (for a different function) with homogeneous boundary condition

Let $v_0 \overset{def}{=} \rho^2 \sin\phi \cos(3\theta)$.

$$f \overset{def}{=} -(\Delta v_0 + v_0)$$

$$u \overset{def}{=} v - v_0$$

The original problem is equivalent to: solve

$$\begin{cases} \Delta u + u = f \\ u\big|_{S^2} = 0 \end{cases} \qquad \text{where} \quad f = -7\sin\phi\cos(3\theta) + 6\cos\varphi - 2 - 9\cos(3\theta)$$

Step 2. Recall the solution of the eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 \\ u\big|_{S^2} = 0 \end{cases}$$

eigenfunctions $\xi_{n,m,i}(\rho, \varphi, \theta) \underset{def}{=} j_n(z_{n,i} \cdot \rho)\, P_n^{|m|}(\cos\varphi)\, e^{\sqrt{-1}\, m\phi}$ $\quad n \in \mathbb{N}$, $m \in \mathbb{Z}, |m| \le n$, $i \in \mathbb{N}$

where $j_n(x) = x^{-\frac{1}{2}} J_{n+\frac{1}{2}}(x)$ are the spherical Bessel functions $\qquad z_{n,1} < z_{n,2} < \cdots$ are the zeros of $j_n(x)$ on $(0,\infty)$

$$\Delta \xi_{n,m,i} + z_{n,i}^2\, \xi_{n,m,i} = 0$$

The eigenfunctions $\xi_{n,m,i}$ form a complete orthogonal family of functions on the ball B_1, and

$$\|\xi_{n,m,i}\|^2 \underset{def}{=} \iiint\limits_{\substack{0 \le \rho \le 1 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi}} |\xi_{n,m,i}(\rho, \varphi, \theta)|^2 \rho^2 \sin\varphi \cdot d\rho\, d\varphi\, d\theta$$

$$\|\xi_{n,m,i}\|^2 = \frac{1}{2}\, j_{n+1}^2(z_{n,i}) \cdot (2\pi) \cdot \frac{4\pi}{2n+1}\, \frac{n+|m|!}{(n-m)!}$$

Step 3. Expand the unknown function u:

$$u = \sum_{n,m,i} A_{n,m,i}\, \xi_{n,m,i}$$

$$\Delta u + u = f \iff \sum_{n,m,i} A_{n,m,i} (1 - z_{n,i}^2)\, \xi_{n,m,i}$$

$$\Rightarrow A_{n,m,i} = \left(\iiint f \cdot \bar{\xi}\, d\rho\, d\varphi\, d\theta\right) \cdot (1 - z_{n,i}^2)^{-1} \cdot j_{n+1}^{-2}(z_{n,i}) \cdot \frac{1}{4\pi^2} (2n+1) \cdot \frac{(n-|m|)!}{(n+|m|)!}$$

Recall that $v = u + u_0$, and u is determined by the coefficients $A_{n,m,i}$ of $\xi_{n,m,i}$