Bessel functions and Legendre polynomials

§1. Special functions: motivations

The designation "special function" is misleading and slightly derogatory; they should really be called "useful functions". Part two of Whittaker–Watson [**?**] contains twelve families of "higher transcendental functions". The updated version [**?**] of Abramowitz–Stegun [**?**] treats 31 families of functions.

The special functions we encounter in math 241 include Bessel functions and Legendre function. We are led to them by the method of *separation of variables*, *the* method for finding explicit solutions of a homogeneous linear PDE with conditions on the boundary of the domain of definition of the (unknown) function. Let's recall the general scheme of this method.

> STEP 0. Choose a suitable coordinate system. (The symmetry properties of the boundary often makes the choice pretty obvious.)
>
> STEP 1. Look for *product solutions*, i.e. solutions which are of the form
> $$u(\xi_1,\dots,\xi_m)=\phi_1(\xi_1)\cdots\phi_m(\xi_m),$$
> where $\xi_1,\dots,\xi_m$ are the coordinates of the (m-dimensional) domain of the unknown function $u(\xi_1,\dots,\xi_m)$. The condition that such a trial product solution satisfies the original linear partial differential equation decouples into m linear ordinary differential equations
> $$P_i(\xi_i,\tfrac{d}{d\xi_i})(\phi_i)=\mu_i,\quad i=1,\dots,m,$$
> where $\mu_1,\dots,\mu_i$ are constants. (The process of "separating the variables $\xi_1,\dots,\xi_m$" usually impose certain obvious relations among the constants $c_1,\dots,c_m$.)
>
> STEP 2. Usually some among the ODE's $P_i(\xi_i,\frac{d}{d\xi_i})(\phi_i)=\mu_i$ have elementary solutions, given by formulas involving rational functions, exponential function, logarithms, and trigonometric functions.
>
> STEP 3. In favorable situations, the general solutions of the rest of the linear ODE's are given by formulas involving special functions.
>
> STEP 4. Often part of the boundary conditions of the original PDE (and the periodicity conditions of the coordinates) gives further constraints on the parameter $\boldsymbol{\mu}=(\mu_1,\dots,\mu_m)$, leaving only a discrete but possibly countably infinite set of permissible parameters $\boldsymbol{\mu}_j$, $j=1,2,\dots$. Each of these parameters $\boldsymbol{\mu_j}$ corresponds to a product solution f_j. So every infinite series $\sum_{j=1}^{\infty} b_j\,f_j$ which is convergent in a suitable sense, satisfies the original linear PDE and part of the boundary conditions.
>
> STEP 5. Choose the coefficients b_j of f_j so that the infinite series $u=\sum_{j=1}^{\infty} b_j\,f_j$ converge and satisfies all boundary conditions.

Special functions appear in step 3 above.

Further readings on PDE.

- If you would like to understand more about partial differential equations, how they arize through the principle of least action and they are tied with spectral theory, you cannot go wrong with volume 1 of Courant–Hilbert [?], a true classic.
- Also recommended is [?], the last of the famous six-term series of lectures by a Nobel laureate in physics.
- For the goal of finding analytic formulas of solutions of linear partial differential equations which appear in many questions in physics and engineering, the standard references are [?] and [?].
- The book [?] covers materials of this course at a higher level.

Further readings on Special functions.

- Whittaker–Watson [?] is a classic in the old British style. Exercises form an integral part of the book; you gain proficiency as you knock them out one by one. Also recommended are the highly readable books [?] and [?].
- The manuscript project [?, ?, ?], Abramowitz–Stegun [?] and its updated version [?] are standard handbook-style references. The latter is available online from the NIST website `https://dlmf.nist.gov`.

References for Bessel functions.

1. For a connected account of the theory of Bessel functions in a book chapter, [?, Ch. 17], [?, §19], [?, Ch 7, §2] and [?, Ch 5] are good sources. Basic complex function theory is freely used in all of them, which you can learn in [?, Ch. 5–6].
2. The handbooks [?, Ch. 7], [?, Ch. 9–10], [?, Ch. 10] are good sources for a quick consultation.
3. The monumental tome [?] with 804 pages is referred to in every treatment of Bessel functions.

§2. Bessel functions

(2.1) The Bessel functions discussed in this note include

- $J_\nu(z)$ (Bessel functions of the first kind), and their close relatives
- $Y_\nu(z)$ (Bessel functions of the second kind, or Weber's function),
- $H_\nu^{(1)}(z), H_\nu^{(2)}(z)$ (Hankel's functions), and

- $I_\nu(z), K_\nu(z)$ (modified Bessel functions, or Bessel functions of imaginary argument).

They are best understood as holomorphic functions indexed by a parameter $\nu \in \mathbb{C}$ on a branched cover of $\mathbb{C}$ branched over the point $z = 0$. In this note we will consider them as holomorphic functions on $\mathbb{C} \smallsetminus (-\infty, 0]$, where their *principal values* are evalued using the *principal branch* of $\log z$ on $\mathbb{C} \smallsetminus (-\infty, 0]$. Recall that the principal branch of $\log z$ on $\mathbb{C} \smallsetminus \{0\}$ so that the phase of the principal values of $\log z$ are $(-\pi, \pi]$ for all $z \neq 0$. For $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ we have

$$\log z = \log |z| + \sqrt{-1}\,\mathrm{ph}(z), \quad \mathrm{ph}(z) \in (-\pi, \pi) \ \ \forall\, z \in \mathbb{C} \smallsetminus (-\infty, 0].$$

Bessel functions of the same order ν are related to each other by

$$H^{(1)}_\nu(z) = J_\nu(z) + \sqrt{-1}\, Y_\nu(z), \quad H^{(2)}_\nu(z) = J_\nu(z) - \sqrt{-1}\, Y_\nu(z) \tag{2.1.1}$$

$$I_\nu(z) = \begin{cases} e^{-\nu\pi\sqrt{-1}/2}\, J_\nu(z\, e^{\pi\sqrt{-1}/2}) & \text{if } -\pi < \mathrm{ph}(z) < \pi/2 \\ e^{\nu\pi\sqrt{-1}/2}\, J_\nu(z\, e^{-\pi\sqrt{-1}/2}) & \text{if } -\pi/2 < \mathrm{ph}(z) < \pi \end{cases} \tag{2.1.2}$$

$$K_\nu(z) = \begin{cases} \frac{\pi\sqrt{-1}}{2}\, e^{\nu\pi\sqrt{-1}/2}\, H^{(1)}_\nu(z\, e^{\pi\sqrt{-1}/2}) & \text{if } -\pi < \mathrm{ph}(z) < \pi/2 \\ -\frac{\pi\sqrt{-1}}{2}\, e^{-\nu\pi\sqrt{-1}/2}\, H^{(2)}_\nu(z\, e^{-\sqrt{-1}/2}) & \text{if } -\pi/2 < \mathrm{ph}(z) < \pi \end{cases} \tag{2.1.3}$$

For every $\nu \in \mathbb{C}$, $J_\nu(z), Y_\nu(z)$ and $H^{(1)}_\nu(z), H^{(2)}_\nu(z)$ are two bases of solutions of Bessel's differential equation of order ν

$$\big((z\tfrac{d}{dz})^2 + z^2\big)u - \nu^2 u = \big(z^2\tfrac{d^2}{dz^2} + z\tfrac{d}{dz} + z^2\big)u - \nu^2 u = 0\,, \tag{2.1.4}$$

while $I_\mu(z), K_\nu(z)$ is a bases of solutions of the modified Bessel equation of order ν

$$\big((z\tfrac{d}{dz})^2 - z^2\big)w - \nu^2 w = \big(z^2\tfrac{d^2}{dz^2} + z\tfrac{d}{dz} - z^2\big)w - \nu^2 w = 0\,. \tag{2.1.5}$$

Remark Equation (2.1.4) for $J_\nu(z), K_\nu(z)$ implies that for every fixed $\nu \in \mathbb{R}$, both $J_\nu(az)$ and $Y_\nu(ax)$ and eigenfunctions for the Sturm–Liouville equation

$$\Big[\tfrac{d}{dx}\big(x\tfrac{d}{dx}\big) - \tfrac{\nu^2}{x} + \lambda x\Big]u = 0 \tag{2.1.6}$$

with eigenvalue $\lambda = a^2$, for every $a \in \mathbb{R}$. Similarly every fixed $\nu \in \mathbb{R}$, both $I_\nu(az)$ and $K_\nu(ax)$ and eigenfunctions for the Sturm–Liouville equation

$$\Big[\tfrac{d}{dx}\big(x\tfrac{d}{dx}\big) + \tfrac{\nu^2}{x} + \lambda x\Big]u = 0 \tag{2.1.7}$$

with eigenvalue $\lambda = a^2$, for every $a \in \mathbb{R}$.

(2.2) Definitions of Bessel functions

The Bessel function of first kind $J_\nu(z)$ is the product of z^ν times an even entire function:

$$J_\nu(z) := (\tfrac{z}{2})^\nu \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{2k}}{k!\,\Gamma(\nu+k+1)!} \qquad \text{if } \nu \notin \mathbb{Z},$$

$$J_n(z) := \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{n+2k}}{k!\,(n+k)!} \qquad \text{if } n \in \mathbb{N}, \tag{2.2.1}$$

$$J_n(z) := \sum_{k=0}^{\infty}(-1)^{k-n} \frac{(z/2)^{-n+2k}}{k!\,(n+k)!} = (-1)^n J_{-n}(z) \qquad \text{if } -n \in \mathbb{N}.$$

Note that if $\nu \notin \mathbb{Z}$, then $J_\nu(z), J_{-\nu}(z)$ is a basis of the solution of Bessel's differential equation (2.1.4). Clearly

$$J_n(-z) = (-1)^n J_n(z) = J_{-n}(z) \quad \forall\, n \in \mathbb{N},\ \forall\, z \in \mathbb{C}. \tag{2.2.2}$$

Recall that the Gamma function $\Gamma(z)$ is a meromorphic function on $\mathbb{C}$ such that $\Gamma(z)^{-1}$ is an entire function which has simple zeros at every non-positive integer and non-zero elsewhere. We have

$$\Gamma(z) = \begin{cases} \int_0^\infty e^{-t} t^z \frac{dt}{t} & \text{if } \mathrm{Re}(z) > 0 \\ \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)} & \text{if} - m < \mathrm{Re}(z) < 0,\ m \in \mathbb{N}_{>0} \end{cases} \tag{2.2.3}$$

for every $z \in \mathbb{C}$ which is not a non-positive integer. The following identities are satisfied.

$$\begin{aligned} \Gamma(z+1) &= z\Gamma(z), \\ \Gamma(z)\,\Gamma(1=z) &= \frac{\pi}{\sin(\pi z)}, \\ 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) &= \sqrt{\pi}\,\Gamma(2z). \end{aligned} \tag{2.2.4}$$

In addition we have

$$\Gamma(n+1) = n! \ \ \forall\, n \in \mathbb{N}, \quad \Gamma(1/2) = \sqrt{\pi}\,. \tag{2.2.5}$$

The functions $Y_\nu(z)$ are defined by

$$\begin{aligned} Y_\nu(z) &:= \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} && \text{if } \nu \notin \mathbb{Z}, \\ Y_n(z) &:= \lim_{\nu \to n} Y_\nu(z) = \frac{1}{\pi}\frac{\partial J_\nu(z)}{\partial \nu}\Big|_{\nu=n} + \frac{(-1)^n}{\pi}\frac{\partial J_\nu(z)}{\partial \nu}\Big|_{\nu=-n} && \text{if } n \in \mathbb{Z}. \end{aligned} \tag{2.2.6}$$

For each $n \in \mathbb{N}$, one gets an explicit series expansion of $Y_n(z)$:

$$\begin{aligned} Y_n(z) = \frac{2}{\pi} J_n(z)\, \log(z/2) - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\Big(\frac{z}{2}\Big)^{2k-n} \\ + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{k!(n+k)!}\big[\psi(k+1) + \psi(k+n+1)\big]\,, \end{aligned} \tag{2.2.7}$$

where

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \tag{2.2.8}$$

is the logarithmic derivative of $\Gamma(z)$. When $n = 0$, the sum $\frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}$ in (2.2.7) is understood to be 0. Note that $z = 0$ is a singular point of $Y_n(z)$ for every $n \in \mathbb{Z}$.

The functions $H_\nu^{(1)}, H_\nu^{(2)}$ are defined by (2.1.1). We also have

$$J_\nu(z) = \tfrac{1}{2}\big(H_\nu^{(1)} + H_\nu^{(2)}\big), \quad Y_\nu(z) = \tfrac{-\sqrt{-1}}{2}\big(H_\nu^{(1)} - H_\nu^{(2)}\big)\,. \tag{2.2.9}$$

The modified Bessel function $I_\nu(z)$ is defined by

$$I_\nu(z) := (\tfrac{z}{2})^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\,\Gamma(\nu+k+1)!} \qquad \forall\, z \in \mathbb{C} \smallsetminus (-\infty, 0] \tag{2.2.10}$$

for all $\nu \in \mathbb{C}$. It extends to an entire function on $\mathbb{C}$ if $\nu \in \mathbb{Z}$:

$$I_n(z) = I_{-n}(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{n+2k}}{k!\,(n+k)!} \qquad \forall\, n \in \mathbb{N}\,. \tag{2.2.11}$$

For $\nu \notin \mathbb{Z}$, the modified Bessel function $K_\nu(z)$ on $\mathbb{C} \smallsetminus (-\infty, 0]$ is defined by

$$K_\nu(z) := \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \quad z \in \mathbb{C} \smallsetminus (-\infty, 0]\,. \tag{2.2.12}$$

For $n \in \mathbb{Z}$, $K_n(z)$ extends to an entire function on $\mathbb{C}$, with

$$K_n(z) := \lim_{\nu \to n} K_\nu(z) \quad \forall\, z \in \mathbb{C}. \tag{2.2.13}$$

It is clear from 2.2.12 that

$$K_{-\nu}(z) = K_\nu(z) \quad \forall\, \nu. \tag{2.2.14}$$

(2.3) Generating functions for $J_\nu(z)$ and $I_\nu(z)$

Many properties of Bessel functions with integral order $J_n(z), I_n(z)$ becomes transparent throught their generating functions:

$$\begin{aligned} e^{z(t-t^{-1})} &= \sum_{n \in \mathbb{Z}} J_n(z) t^n, \\ e^{z(t+t^{-1})} &= \sum_{n \in \mathbb{Z}} I_n(z) t^n. \end{aligned} \tag{2.3.1}$$

(2.4) Bessel functions of half-integral order

The Bessel functions $J_\nu(z), Y_\nu(z), H_\nu^{(1)}(z), H_\nu^{(2)}(z), I_\nu(z), K_\nu(z)$ can be expressed in terms of elementary functions when ν is a half-integer. For example

$$J_{1/2}(z) = \left(\tfrac{2}{\pi z}\right)^{1/2} \sin z, \tag{2.4.1}$$

$$J_{-1/2}(z) = \left(\tfrac{2}{\pi z}\right)^{1/2} \cos z. \tag{2.4.2}$$

Using recurrence relations we can get expressions for all $J_\nu(z)$ with $\nu = n+\frac{1}{2}$ for some $n \in \mathbb{N}$. For instance

$$J_{n+\frac{1}{2}}(z) = (-1)^n \left(\tfrac{2}{\pi}\right)^{1/2} z^{n+\frac{1}{2}} \left(z^{-1}\tfrac{d}{dz}\right)^n \frac{\sin z}{z}, \quad n = 0, 1, 2, \ldots \tag{2.4.3}$$

(2.5) Recurrence relations

Denote by $\mathscr{C}_\nu(z)$ any of the four Bessel functions $J_\nu(z), Y_\nu(z), H_\nu^{(1)}(z), H_\nu^{(1)}(z)$. We have

$$\mathscr{C}_{\nu-1}(z) + \mathscr{C}_{\nu+1}(z) = \frac{2\nu}{z}\mathscr{C}_\nu(z), \tag{2.5.1}$$

$$\begin{aligned} \mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z) &= 2\,\tfrac{d}{dz}\mathscr{C}_\nu(z) \quad \text{if } \nu \neq 0 \\ -\mathscr{C}_1(z) &= \tfrac{d}{dz}\mathscr{C}_0(z) \end{aligned} \tag{2.5.2}$$

$$\left(z^{-1}\tfrac{d}{dz}\right)\left[z^\nu \mathscr{C}_\nu(z)\right] = z^{\nu-1}\mathscr{C}_{\nu-1}(z) \tag{2.5.3}$$

$$\left(z^{-1}\tfrac{d}{dz}\right)\left[z^{-\nu} \mathscr{C}_\nu(z)\right] = -z^{-\nu-1}\mathscr{C}_{\nu+1}(z)\,. \tag{2.5.4}$$

The functions $I_\nu(z), K_\nu(z)$ satisfy similar recurrence relations with different signs at places.

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z), \quad I_{\nu-1}(z) + I_{\nu+1}(z) = 2\,\tfrac{d}{dz} I_\nu(z), \tag{2.5.5}$$

$$\left(z^{-1}\tfrac{d}{dz}\right)\left[z^\nu I_\nu(z)\right] = z^{\nu-1} I_{\nu-1}(z), \quad \left(z^{-1}\tfrac{d}{dz}\right)\left[z^{-\nu} I_\nu(z)\right] = z^{-\nu-1} I_{\nu+1}(z)\,. \tag{2.5.6}$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z), \quad K_{\nu-1}(z) + K_{\nu+1}(z) = -2\,\tfrac{d}{dz} K_\nu(z), \tag{2.5.7}$$

$$\left(z^{-1}\tfrac{d}{dz}\right)\left[z^\nu K_\nu(z)\right] = -z^{\nu-1} K_{\nu-1}(z), \quad \left(z^{-1}\tfrac{d}{dz}\right)\left[z^{-\nu} K_\nu(z)\right] = -z^{-\nu-1} K_{\nu+1}(z)\,. \tag{2.5.8}$$

(2.6) Connection formulas

Besides (2.1.1)–(2.1.3), we have

$$\mathscr{C}_{-n}(z) = (-1)^n \mathscr{C}_n(z) \tag{2.6.1}$$

for all $n \in \mathbb{Z}$, where $\mathscr{C}_n(z)$ denote any one of $J_n(z), Y_n(z), H_n^{(1)}(z), H_n^{(2)}(z)$ as in 2.5, and

$$\begin{aligned} H_{-\nu}^{(1)}(z) &= e^{\nu\pi\sqrt{-1}} H_\nu^{(1)}(z), \\ H_{-\nu}^{(2)}(z) &= e^{-\nu\pi\sqrt{-1}} H_\nu^{(2)}(z). \end{aligned} \tag{2.6.2}$$

(2.7) Asymptotics of Bessel functions as $z \to 0$

We see from (2.2.1) and (2.2.10) that

$$J_n(z) = (-1)^n J_{-n}(z) \ \sim \ \frac{z^n}{2^n \cdot n!} \qquad \text{as } z \to 0,\ n \in \mathbb{N}\,, \tag{2.7.1}$$

and

$$I_n(z) = I_{-n}(z) \ \sim \ \frac{z^n}{2^n \cdot n!} \qquad \text{as } z \to 0,\ n \in \mathbb{N}\,. \tag{2.7.2}$$

Similarly the series expansion (2.2.7) implies that

$$Y_0(z) \ \sim \ \tfrac{2}{\pi}\, \log z \quad \text{as } z \to 0\,, \tag{2.7.3}$$

$$Y_n(z) = (-1)^n Y_{-n}(z) \ \sim \ -\tfrac{2^n (n-1)!}{\pi}\, z^{-n} \quad \text{as } z \to 0,\ n \in \mathbb{N}_{\geq 1}\,, \tag{2.7.4}$$

and (2.2.13) implies that

$$K_0(z) \ \sim \ \log(\tfrac{2}{z}) \quad \text{as } z \to 0\,, \tag{2.7.5}$$

$$K_n(z) = K_{-n}(z) \ \sim \ (2^{n-1}(n-1)!) \cdot z^{-n} \quad \text{as } z \to 0,\ n \in \mathbb{N}_{\geq 1}\,. \tag{2.7.6}$$

(2.8) Integral representations of Bessel functions

Bessel functions have many integral representations, each with some restriction on the pair (ν, z). We give a few examples. First we have

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin\theta - n\theta)\, d\theta, \quad n \in \mathbb{Z}, \tag{2.8.1}$$

For z with $\mathrm{Re}(z) > 0$, we have contour integral representations

$$J_\nu(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\infty - \pi\sqrt{-1}}^{\infty + \pi\sqrt{-1}} e^{z \sinh t - \nu t}\, dt, \tag{2.8.2}$$

and

$$\begin{aligned} H_\nu^{(1)} &= \frac{1}{\pi\sqrt{-1}} \int_{-\infty}^{\infty + \pi\sqrt{-1}} e^{z \sinh t - \nu t}\, dt, \\ H_\nu^{(2)} &= -\frac{1}{\pi\sqrt{-1}} \int_{-\infty}^{\infty - \pi\sqrt{-1}} e^{z \sinh t - \nu t}\, dt. \end{aligned} \tag{2.8.3}$$

(2.9) Asymptotic behavior of Bessel functions as $z \to \infty$

When ν is fixed and $z \to \infty$, for every $\delta > 0$ we have

$$\begin{aligned}
J_\nu(z) &= \sqrt{2/(\pi z)}\,\Big(\cos\big(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big) + e^{|\mathrm{Im}(z)|}o(1)\Big),\\
Y_\nu(z) &= \sqrt{2/(\pi z)}\,\Big(\sin\big(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big) + e^{|\mathrm{Im}(z)|}o(1)\Big),\\
H_\nu^{(1)}(z) &= \sqrt{2/(\pi z)}\, e^{\sqrt{-1}\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}\big(1+o(1)\big),\\
H_\nu^{(2)}(z) &= \sqrt{2/(\pi z)}\, e^{-\sqrt{-1}\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}\big(1+o(1)\big),
\end{aligned} \tag{2.9.1}$$

uniformly for all $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ with $|\mathrm{ph}(z)| \le \pi - \delta$.

The asymptotic behavior of the modified Bessel functions $I_\nu(z), K_\nu(z)$ are obtained from (2.9.1) by substitutions via (2.1.2) and (2.1.3) respectively. In particular we have

$$\begin{aligned}
I_\nu(x) &= \tfrac{1}{2\pi x}\, e^{x}\,\big(1+o(1)\big),\\
K_\nu(x) &= \tfrac{\pi}{2x}\, e^{-x}\,\big(1+o(1)\big),
\end{aligned} \quad \text{as } x \to \infty,\ x \in \mathbb{R}. \tag{2.9.2}$$

for every $\nu \ge 0$.

(2.10) Zeroes of Bessel functions $J_\nu(z), J'_\nu(z), Y_\nu(z), Y'_\nu(z)$ with $\nu \in \mathbb{R}$

The zeros of any Bessel function or its derivative are simple, with the possible exception of $z = 0$, because the Bessel differential equation is regular on $\mathbb{C} \smallsetminus \{0\}$.

When $\nu \in \mathbb{R}$, each of $J_\nu(z), J'_\nu(z), Y_\nu(z), Y'_\nu(z)$ has an infinite number of positive zeros. Let

$$j_{\nu,1} < j_{\nu,2} < j_{\nu,3} \cdots, \quad j'_{\nu,1} < j'_{\nu,2} < j'_{\nu,3} < \cdots$$

be the list of positive zeros of $J_\nu(z)$ and $J'_\nu(z)$ respectively, except that when $\nu = 0$, we count $z = 0$ as the first "positive zero" of $J'_0(z)$, so that

$$j'_{0,1} = 0, \quad j'_{0,m} = j_{1,m} \ \text{ for } m = 1, 2, 3, \ldots.$$

Similarly let

$$y_{\nu,1} < y_{\nu,2} < y_{\nu 3} \cdots, \quad y'_{\nu,1} < y'_{\nu,2} < y'_{\nu,3} < \cdots$$

be the list of positive zeros of $Y_\nu(z)$ and $Y'_\nu(z)$ respectively

The positive zeros of $J_\nu(z)$ and $J_{\nu+1}(z)$ interlace, and so do the positive zeros of $Y_\nu(z)$ and $Y_{\nu+1}(z)$, for every $\nu \ge 0$:

$$\begin{aligned}
j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \cdots,\\
y_{\nu,1} < y_{\nu+1,1} < y_{\nu,2} < y_{\nu+1,2} < y_{\nu,3} < \cdots.
\end{aligned} \tag{2.10.1}$$

The positive zeros of $J'_\nu(z), Y_\nu(z), Y'_\nu(z), J_\nu(z)$ with the same order $\nu \geq 0$ interlace according to the following inequalities:

$$\nu \leq j'_{\nu,1} < y_{\nu,1} < y'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < y_{\nu,2} < y'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \cdots . \tag{2.10.2}$$

It follows from (2.9.1) that

$$\begin{aligned} j_{\nu,n} &= \pi n(1+o(1)) \quad \text{as } n \to \infty, \\ y_{\nu,n} &= \pi n(1+o(1)) \quad \text{as } n \to \infty, \end{aligned} \tag{2.10.3}$$

for every $\nu \geq 0$. The inequalities (2.10.2) implies that

$$\begin{aligned} j'_{\nu,n} &= \pi n(1+o(1)) \quad \text{as } n \to \infty, \\ y'_{\nu,n} &= \pi n(1+o(1)) \quad \text{as } n \to \infty \end{aligned} \tag{2.10.4}$$

as well.

Remark (a) If $\nu \geq -1$, then all zeros of $J_\nu(z)$ are real. If $\nu \geq 0$, then all zeros of $J'_\nu(z)$ are real.
(b) If $\nu < -1$ and ν is not an integer, $J_\nu(z)$ has exactly $\lfloor -\nu \rfloor$ pairs of complex conjugate zeros.

(2.11) Orthogonality relations

Let $\nu \geq -\frac{1}{2}$ be a real number. For any two distinct non-zero real numbers α, β, the functions $u_\alpha(x) := J_\nu(\alpha x), u_\beta(x) := J_\nu(\beta x)$ satisfy

$$\tfrac{d}{dx}\big(x(u_\alpha u'_\beta - u_\beta u'_\alpha)\big) = (\alpha^2 - \beta^2)\, x u_\alpha u_\beta,$$

and we get

$$\int_0^1 x J_\nu(\alpha x) J_\nu(\beta x)\, dx = \frac{\beta J_\nu(\alpha) J'_\nu(\beta) - \alpha J_\nu(\beta) J'_\nu(\alpha)}{\alpha^2 - \beta^2}. \tag{2.11.1}$$

Let α, β be two distinct positive zeros $j_{\nu,m}$ of $J_\nu(x)$, it follows that

$$\int_0^1 x J_\nu(j_{\nu,m} x) J_\nu(j_{\nu,n} x)\, dx = 0 \quad \text{if } m \neq n. \tag{2.11.2}$$

Take the limit as $\beta \to \alpha$ in (2.11.1) using L'Hospital's rule and eliminate J''_ν with Bessel's differential equation, we get

$$\int_0^1 x J_\nu^2(j_{\nu,n} x)\, dx = \tfrac{1}{2}\,(J'_\nu(j_{\nu,n}))^2 = \tfrac{1}{2}\, J_{\nu+1}^2(j_{\nu,n}). \tag{2.11.3}$$

(2.11.4) Proposition (Fourier–Bessel expansion) *Let $\nu \geq -1/2$ be a real number. Let $f(x)$ be a piecewise continuous function on $(0,1)$ such that*

$$\int_0^1 x^{1/2}\,|f(x)|\,dx < \infty$$

and $f(t)$ is of bounded variation in every interval $[a,b]$ with $0 < a < b < 1$. Define numbers c_m by

$$c_m := \frac{2}{J^2_{\nu+1}(j_{\nu,m})}\int_0^1 x\,f(x)\,J_\nu(j_{\nu,m}x)\,dx, \quad m \geq 1.$$

Then

$$\tfrac{1}{2}\big(f(x-)+f(x+)\big) = \sum_{m=1}^{\infty} c_m\,J_\nu(j_{\nu,m}x)\,.$$

(2.11.5) Proposition (Fourier–Bessel integral) *Let $\nu \geq -1/2$ be a real number. Let $f(x)$ be a piecewise continuous funtion on $(0,\infty)$ and of bounded variation on every finite subinterval $[a,b]$ with $0 < a < b < \infty$. Assume that*

$$\int_0^\infty x^{1/2}\,|f(x)|\,dx < \infty.$$

Then for every $x \in (0,\infty)$ we have

$$\tfrac{1}{2}\big(f(x-)+f(x+)\big) = \int_0^\infty \lambda J_\nu(\lambda x)\,d\lambda \int_0^\infty \rho J_\nu(\rho x) f(\rho)\,d\rho\,.$$

§3. Legendre functions and spherical harmonics

(3.1) The *Legendre polynmials* $P_n(x)$ form one of many families of *orthogonal polynomials.* We can use $P_n(x)$ and its associated *Legendre functions* $P_n^m(x)$, $m \in \mathbb{N}$, to write down an explicit basis of harmonic polynomials homogeneous of degree n in 3 variables, i.e. polynomials $f(x,y,z)$ homogeneous of degree n such that $\Delta f = 0$.

There are more general Legendre functions $P_\nu(z), Q_\nu(z)$ where the parameter ν can be any complex number, and their associated Legendre functions $P_\nu^m(z), Q_\nu^m(z)$, $m \in \mathbb{N}$. They are also called *spherical harmonics* in the literature.

Perhaps the best motivation of Legendre polynomials is through their generating functions. We will use vector notations in $\mathbb{R}^3$, and let $\mathbf{r} = (x,y,z)$, and let $\mathbf{r}_0 = (x_0,y_0,z_0)$ be a fixed point in $\mathbb{R}^3$ which is different from the origin. It is well-known that $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$ is a harmonic function on $\mathbb{R}^3 \smallsetminus \{0\}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{r}_0$, let $t := \frac{|\mathbf{r}|}{|\mathbf{r}_0|}$, and let ϕ be the angle of (x,y) so that

$$x = |\mathbf{r}_0|\,t\,\sin\theta\cos\phi, \quad y = |\mathbf{r}_0|\,t\,\sin\theta\sin\phi, \quad z = |\mathbf{r}_0|\,t\,\cos\theta,$$

and

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \frac{1}{|\mathbf{r}_0|}\cdot\frac{1}{1-2t\cos\theta+t^2}. \tag{3.1.1}$$

The Laplacian in coordintes[1] (t,θ,ϕ) is

$$\triangle = \frac{1}{|\mathbf{r}_0|^2\, t^2\, \sin\theta}\cdot\Big[\frac{\partial}{\partial t}\big(t^2\,\frac{\partial}{\partial t}\big) + \frac{\partial}{\partial\theta}\big(\sin\theta\,\frac{\partial}{\partial\theta}\big) + \frac{\partial}{\partial\phi}\big(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\big)\Big] \tag{3.1.2}$$

Let $w := \cos\theta$. In the coordinate system (t,w,ϕ), we have

$$\frac{\partial}{\partial\theta} = -\sin\theta\,\frac{\partial}{\partial w}, \qquad \frac{\partial^2}{\partial\theta^2} = -w\frac{\partial}{\partial w} + (1-w^2)\frac{\partial}{\partial w^2}.$$

Define polynomials $P_n(w)$, $n\in\mathbb{N}$, by

$$\frac{1}{1-2tw+t^2} = \sum_{n=0}^{\infty} P_n(w)t^n. \tag{3.1.3}$$

The fact that $\triangle\frac{1}{1-2tw+t^2} = 0$ translates into

$$\Big[\big((1-w^2)\tfrac{d}{dw}\big)^2 + n(n+1)(1-w^2)\Big]P_n(w) = 0, \quad \forall\, n\in\mathbb{N}. \tag{3.1.4}$$

References for Legendre functions: [**?**, Ch. 4 & Ch. 8], [**?**, Ch. 10], [**?**, Ch. 14], [**?**, Ch. 3], [**?**, Ch. 7, §§3–5], [**?**, Ch. 14].

(3.2) Definitions and the differential equations they satisfy

The Legendre polynomials can be defined either directly, or via their generating function $(1-2xt+t^2)^{-1/2}$.

$$\begin{aligned} P_n(x) &:= \tfrac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n \\ &= \frac{1}{2^n\, n!}\sum_{k=\lceil n/2\rceil}^{n}(-1)^{n-k}\binom{n}{k}\frac{(2k)!}{(2k-n)!}\,x^{2k-n} \end{aligned} \tag{3.2.1}$$

$$(1-2xt+t^2)^{-1/2} = \sum_{n\geq 0} P_n(x)\,t^n\,.$$

The parameter n here is called either the *degree* of the *order*.

The associated Legenedre functions of integer order n are defined by

$$P_n^m(x) := (1-x^2)^{m/2}\,\tfrac{d^m}{dx^m}P_n(z) = (1-x^2)^{m/2}\,\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n, \quad m\in\mathbb{N} \tag{3.2.2}$$

The superscrip m in $P_n^m(x)$ is a parameter, and $P_n^m(x)$ is *not* the m-th power of $P_n(x)$. Clearly

$$P_n^0(x) = P_n(x), \quad P_n^m(x) = 0 \ \ \forall\, m > n. \tag{3.2.3}$$

[1] Here we have followed the convention that θ denotes the angle to the norht-south pole, as in the majority of literature in physics in Engineering. However in the most calculus textbooks, this angle is denoted by ϕ.

EXAMPLES.

$$P_0(x) = 1, \qquad P_1(x) = x,$$
$$P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}, \qquad P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x,$$
$$P_4(x) = \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8}, \qquad P_5(x) = \tfrac{63}{8}x^5 - \tfrac{35}{4}x^3 + \tfrac{15}{8}x.$$

It is clear that $\deg(P_n(x)) = n$ and $P_n(-x) = (-1)^n P_n(x)$, for every $n \in \mathbb{N}$.

The Legendre polynomial $P_n(x)$ is a solution of the differential equation

$$\frac{d}{dx}\Big((1-x^2)\,\frac{du}{dx}\Big) + n(n+1)u = 0, \tag{3.2.4}$$

or equivalently

$$\big[\big((1-x^2)\,\tfrac{d}{dx}\big)^2 + n(n+1)(1-x^2)\big]u = 0, \tag{3.2.5}$$

while the associated Legendre function $P_n^m(x)$ is a solution of the differential equatin

$$\big[\big((1-x^2)\tfrac{d}{dx}\big)^2 + n(n+1)(1-x^2) - m^2\big]u = 0. \tag{3.2.6}$$

In other words, the Legendre function $P_n^m(x)$ with $m, n \in \mathbb{N}$ and $0 \leq m \leq n$ is an eigenfunction for the Sturm–Liouville equation

$$\Big[\tfrac{d}{dx}\big((1-x^2)\tfrac{d}{dx}\big) - \tfrac{m^2}{1-x^2} + \lambda\Big]\, u = 0 \tag{3.2.7}$$

with eigenvalue $\lambda = n(n+1)$.

(3.3) Recurrence relations

Taking the logarithmic derivative of the generating function $(1-2xt+t^2)^{-1/2}$ of the Legendre polynomials with respect to t and to x leads to a number of recurrence relations.

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad \forall\, n \in \mathbb{N}, \tag{3.3.1}$$

$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) = P_n(x) \quad \forall\, n \geq 1, \tag{3.3.2}$$

$$\begin{aligned} P'_{n+1}(x) - xP'_n(x) &= (n+1)P_n(x) \quad \forall\, n \geq 0, \\ xP'_n(x) - P'_{n-1}(x) &= nP_n(x) \quad \forall\, n \geq 1, \\ P'_{n+1}(x) - P'_{n-1}(x) &= (2n+1)P_n(x) \quad \forall\, n \geq 1, \end{aligned} \tag{3.3.3}$$

$$\Big((1-x^2)\frac{d}{dx}\Big)P_n(x) = nP_{n-1}(x) - nxP_n(x) \quad \forall\, n \geq 1. \tag{3.3.4}$$

(3.4) Orthogonality relations

The Legendre polynomials are mutually orthogonal as elements of the Hilbert space $L^2([-1,1])$ consisting of all square integrable functions on $[-1,1]$.

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0 & \text{if } \; m \neq n, \\ \frac{2}{2n+1} & \text{if } \; m = n. \end{cases} \tag{3.4.1}$$

The fact that $\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0$ for $m \neq n$ is a special case of the general orthogonality relation between eigenfunctions assocated to different eigenvalues in a Sturm–Liouville differential equation.

Proof. We start from the differential equation (3.2.4) for $P_n(x)$ and $P_m(x)$. An easy calculation yields

$$(n-m)(n+m+1)P_n(x)P_m(x) = \tfrac{d}{dx}\big[(1-x^2)(P_m'(x)P_n(x) - P_m(x)P_n'(x))\big], \tag{3.4.2}$$

which implies that

$$(n-m)(n+m+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0.$$

To show that $\int_{-1}^{1} P_n(x)^2\, dx = \frac{2}{2n+1}$, from the generating function for the $P_n(x)$'s, we get

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)\, dx \cdot t^{m+n} = \sum_{n=0}^{\infty}\int_{-1}^{1} P_n(x)^2\, dx \cdot t^{2n}.$$

An elementary calculation shows that

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = -\frac{1}{2t}\log\tfrac{(1-t)^2}{(1+t)^2} = \sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1}. \quad \square$$

More generally, we have the following orthogonality relations for Legendre functions.

$$\int_{-1}^{1} P_l^m(x)P_n^m(x)\, dx = \delta_{l,n}\frac{(n+m)!}{(n-m)!\,(n+\frac{1}{2})}\,, \tag{3.4.3}$$

$$\int_{-1}^{1} \frac{P_n^l(x)P_n^m(x)}{1-x^2}\, dx = \delta_{l,m}\frac{(n+m)!}{(n-m)!\,m}\,. \tag{3.4.4}$$

(3.4.5) Proposition *For every $n \in \mathbb{N}$, the polynomial $P_n(x)$ of degree n has n distinct zeros in the open interval $(-1,1)$.*

This proposition is a special case of a general fact about orthogonal polynomials.

(3.4.6) Proposition *The orthogonal polynomials* $\{P_n(x) \mid n \in \mathbb{N}\}$ *form a* complete *orthogonal basis of* $L^2([-1,1])$. *In other words for every square integralbe* $\mathbb{R}$*-valued function* $f(x) \in L^2([-1,1])$, *the equality*

$$f(x) = \sum_{n\in\mathbb{N}} P_n(x) \cdot \tfrac{2n+1}{2} \int_{-1}^{1} f(t)\, P_n(t)\, dt$$

holds in $L^2[-1,1]$ *in the sense that*

$$\lim_{N\to\infty} \int_{-1}^{1} \Big[f(x) - \sum_{n=0}^{N} P_n(x) \cdot \tfrac{2n+1}{2} \int_{-1}^{1} f(t)\, P_n(t)\, dt \Big]^2 dx = 0.$$

(3.5) Integral representations of Legendre polynomials

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2-1})^n \, d\phi. \qquad (3.5.1)$$

$$P_n(\cos\theta) = \frac{2}{\pi} \int_0^{\theta} \frac{\cos(n+\frac{1}{2})\psi}{\sqrt{2\cos\psi - 2\cos\theta}}\, d\psi \quad 0 < \theta < \pi,\ n \in \mathbb{N}. \qquad (3.5.2)$$

The following bounds on the values of $P_n(x)$ can be established with the help of the integral representations above.

(3.5.3) Proposition *For every* $x \in (-1,1)$ *and every integer* $n \geq 1$, *we have*

$$|P_n(x)| < 1$$

and also

$$|P_n(x)| < \Big[\frac{\pi}{2n(1-x^2)}\Big]^{1/2}.$$

(3.6) Asymptotic representation of Legendre polynomials $P_n(x)$ for $n \gg 0$

$$P_n(\cos\theta) \sim \sqrt{\tfrac{2}{\pi n \sin\theta}}\, \sin\big[(n+\tfrac{1}{2})\theta + \tfrac{\pi}{4}\big], \quad n \to \infty,\ \delta \leq \theta \leq \pi - \delta \qquad (3.6.1)$$

for any $\delta > 0$. In other words, for every $\delta > 0$ and every $\epsilon > 0$, there exists a natural number N_0 such that

$$-\epsilon < \frac{P_n(\cos\theta)\,\sqrt{\pi n \sin\theta}}{\sqrt{2}\,\sin\big[(n+\frac{1}{2})\theta + \frac{\pi}{4}\big]} - 1 < \epsilon \qquad \forall n \geq N_0,\ \forall\, \theta \in [\delta, \pi-\delta].$$

(3.7) Spherical functions

Let $\triangle_{S^2}$ be the Laplacian on the unit sphere. In spherical coordinate (θ, ϕ), $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $\triangle_{S^2}$ is given by

$$\triangle_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\,. \tag{3.7.1}$$

The eigenvalues of the equation

$$(\triangle_{S^2} + \lambda)u = 0$$

are

$$\lambda = n \cdot (n+1), \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, a basis of the solutions of

$$(\triangle_{S^2} + n(n+1))u = 0$$

is given by $Y_n^k(\theta, \phi)$, $k = 0, \pm 1, \ldots, \pm n$,

$$Y_n^k(\theta,\phi) = \begin{cases} P_n^k(\cos\theta)\cos(k\phi) & \text{if } \ k \geq 0, \\ P_n^{-k}(\cos\theta)\sin(-k\phi) & \text{if } \ k < 0. \end{cases} \tag{3.7.2}$$

The spherical functions Y_n^k with $n \in \mathbb{N}$ and $|k| \leq n$ form an orthogonal basis of the Hilbert space $L^2(S^2)$ of all square integrable functions on the unit sphere S^2, and we get from 3.4.3 that

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} Y_n^k(\theta,\phi)^2 \,\sin\theta\, d\phi\, d\theta = \begin{cases} \frac{4\pi}{2n+1} & \text{if } \ k = 0, \\ \frac{2\pi}{2n+1}\,\frac{(n+|k|)!}{(n-|k|)!} & \text{if } \ 1 \leq |k| \leq n. \end{cases} \tag{3.7.3}$$