Eigenfunction of Laplacian on disks, spheres and balls

1. Eigenvalue problems for the Laplace operator on the uni disk

$$\Delta_{D_1} = r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + r^{-2}\frac{\partial^2}{\partial\theta^2}, \quad \Delta_{D_{\le 1}} u + \lambda u = 0 \qquad D_{\le 1} = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$$

1a. Dirichlet boundary condition : $u|_{\partial D_1} = 0$

eigenfunctions : $J_n(j_{n,i}\cdot\rho)\, e^{\pm\sqrt{-1}n\theta}$ $\quad n = 0, 1, 2, \dots$ $\quad 0 < j_{n,1} < j_{n,2} < \cdots$ positive zeros of $J_n(x)$

eigenvalue for $J_n(j_{n,i}\cdot\rho)\, e^{\pm\sqrt{-1}n\theta}$: $j_{n,i}^2$

i.e. $\left(\Delta_{D_1} + j_{n,i}^2\right)\left(J_n(j_{n,i}\cdot\rho)\, e^{\pm\sqrt{-1}n\theta}\right) = 0$

1b. Neumann boundary condition : $\left.\frac{\partial u}{\partial r}\right|_{\partial D} = 0$

eigenfunctions : $J_n(j'_{n,i}\cdot\rho)\, e^{\pm\sqrt{-1}n\theta}$ $\quad n = 0, 1, 2, \dots$ $\quad 0 < j'_{n,1} < j'_{n,2} < \cdots$ positive zeros of $\frac{d}{dx}J_n(x)$

eigenvalue for $J_n(j'_{n,i}\cdot\rho)\, e^{\pm\sqrt{-1}n\theta}$: ${j'_{n,i}}^2$

2. Eigenvalue problems for the Laplace operator on the 2-dimensional unit sphere S^2

$$\Delta_{S^2} = (\sin\varphi)^{-1}\left[\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{\partial}{\partial\theta}(\sin\varphi)^{-1}\frac{\partial}{\partial\theta}\right]$$

$0\le\varphi\le\pi$ co-altitude

$0\le\theta\le 2\pi$ azimuthal

eigenvalues : $n(n+1)$, $n = 0, 1, 2, 3, \dots$

eigenfunctions with eigenvalue $n(n+1)$:

$$\begin{cases} P_n^k(\cos\varphi)\cdot\cos(k\theta), & k = 0, 1, \dots, n \\ \text{and } P_n^k(\cos\varphi)\sin(k\theta) & k = 1, \dots, n \end{cases}$$

equiv. $P_n^{|k|}(\cos\varphi)\, e^{\sqrt{-1}k\theta}$, $-n\le k\le n$

Applications

2a. Harmonic functions on the a ball $B_{\le a} = \{(x,y,z)\in\mathbb{R}^3 \mid x^3+y^3+z^3\le a^2\}$, $a>0$

an ortho basis of harmonic functions on $B_{\le a}$

$\rho^n\cdot P_n^m(\cos\varphi)\cdot\cos(m\theta)$ $\quad n = 0, 1, 2, \dots$ $\quad m = 0, 1, \dots, n$

and $\rho^n P_n^m(\cos\varphi)\cdot\sin(m\theta)$ $\quad n = 0, 1, 2, \dots$ $\quad m = 1, 2, 3, \dots$

2b. Harmonic functions on $B_{\geq a} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \geq a^2\}$, $a>0$ which are bounded on $B_{\geq a}$

An orthogonal basis:

$$\rho^{-n-1} P_n^m(\cos\varphi)\cos(m\theta), \quad n=0,1,2,\dots, \quad m=0,1,\dots,n$$

$$\rho^{-n-1} P_n^m(\cos\varphi)\sin(m\theta), \quad n=0,1,2,\dots, \quad m=0,1,\dots,n$$

2c. Laplace operator on the unit ball $B_{\leq 1}$ with Dirichlet boundary condition

eigenfunctions: $\rho^{-\frac{1}{2}} \cdot J_{n+\frac{1}{2}}(j_{n+\frac{1}{2},i} \cdot \rho) \cdot P_n^m(\cos\varphi) \cdot \begin{cases} \cos(m\theta) & m=0,1,2,\dots,n \\ \sin(m\theta) & m=1,2,3,\dots,n \end{cases}$

$\Delta_{B_1} u + \lambda u = 0$ $\quad n=0,1,2,\dots,$

$i=1,2,3,\dots$

eigenvalue $= j_{n+\frac{1}{2}}^2$

where $0 < j_{n+\frac{1}{2},1} < j_{n+\frac{1}{2},2} < \dots < j_{n+\frac{1}{2},k} < \dots$ are the positive zeros of $J_{n+\frac{1}{2}}(x)$

2d. Laplace operator on the unit ball $B_{\leq 1}$ with Neumann boundary condition

eigenvalues $(j'_{n+\frac{1}{2},i})^2$, $n=0,1,2,\dots$ where $0 < j'_{n+\frac{1}{2},1} < j'_{n+\frac{1}{2},2} < \dots$ are the positive zeros of $\frac{d}{dx}\mathbb{j}_n(x)$

eigenfunctions with eigenvalue $(j'_{n+\frac{1}{2},i})^2$:

$$\rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(j'_{n+\frac{1}{2},i} \cdot \rho) \cdot P_n^m(\cos\varphi) \cdot \begin{cases} \cos(m\theta) & m=0,1,2,\dots,n \\ \sin(m\theta) & m=1,2,\dots,n \end{cases}$$

Note: $\mathbb{j}_n(z) \overset{\text{def}}{:=} \sqrt{\frac{\pi}{2}} \cdot z^{-\frac{1}{2}} J_{n+\frac{1}{2}}(z) = (-1)^n \sqrt{\frac{\pi}{2}}\, z^{-\frac{1}{2}} Y_{-n-\frac{1}{2}}(z)$

and $y_n(z) := \sqrt{\frac{\pi}{2}}\, z^{-\frac{1}{2}} Y_{n+\frac{1}{2}}(z) = (-1)^{n+1} \sqrt{\frac{\pi}{2}}\, z^{-\frac{1}{2}} J_{-n-\frac{1}{2}}(z)$ $\quad n=0,1,2,3,\dots$

are called the spherical functions of the first kind

The condition that $\frac{d}{d\rho}\left(\rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda\rho)\right)\Big|_{\rho=1}$ is exactly that $\frac{d}{dx}\mathbb{j}_n(x)\Big|_{x=\lambda} = 0$