Supplements on Fourier series

§1. Convergence theorems and the Parseval identity

(1.1) Periodic functions as functions on $\mathbb{R}/2L\mathbb{Z}$.

Let $L > 0$ be a positive integer, and we will consider periodic functions with period $2L$ on $\mathbb{R}$, i.e. functions $g(x)$ on $\mathbb{R}$ such that $g(x + 2L) = g(x)$ for all $x \in \mathbb{R}$. If we identify any two points of $\mathbb{R}$ which differ from each other by an integer multiple of $2L$, the resulting space $\mathbb{R}/2L\mathbb{Z}$ is essentially the same as a circle. The function $x \mapsto e^{\frac{\pi\sqrt{-1}x}{L}}$ gives an explicit identification. Every periodic function with period $2L$ is really a function on $\mathbb{R}/2L\mathbb{Z}$.

Given any $a \in \mathbb{R}$, the corresponding point on $\mathbb{R}/2L\mathbb{Z}$ will be denoted by $a \bmod 2L$. Note that for every $b \in \mathbb{R}$ such that $b - a$ is an integer multiple of $2L$, $b \bmod 2L$ is equal to $a \bmod 2L$. If $b - a$ is an integer multiple of $2L$, we say that a and b are *congruent modulo* $2L$ and write $b \equiv a \pmod{2L}$ for this relation.

Point-wise convergence of Fourier series is a complicated business, The Fourier series associated to a continuous periodic function may *not* converge; there are examples of a continuous periodic function $f(x)$ whose Fourier series diverges at *uncoutably many* points. However all functions you will meet in math 241 are *piece-wise smooth* (but not necessarily continuous). I will state two propositions for your peace of mind.

(1.2) Definition A function defined on a finite interval $[a, b]$, $a < b$ is said to be *of bounded variation* if there exists a number $C > 0$ such that for every finite sequence $a \le x_1 < x_2 < \cdots < x_m \le b$ we have

$$\sum_{k=1}^{m-1} |f(x_{m+1}) - f(x_m)| < C.$$

A periodic function of period $2L$, $L > 0$, is said to be of bounded variation if its restriction to a period (e.g. $[-L, L]$) is of bounded variation.

(1.3) Remark (a) It is obvious that every piece-wise continuously differentiable function on a finite interval $[a, b]$ is of bounded variation. A fortiori, every piece-wise smooth function on a finite interval is of bounded variation.

(b) It is equally obvious that every non-decreasing $\mathbb{R}$-valued function on $[a, b]$ is of bounded variation. It is equally clear that every non-increasing $\mathbb{R}$-valued function on $[a, b]$ is of bounded variation.

(c) It is a fact that every function f of bounded variation on a finite interval $[a, b]$ is a sum of a non-decreasing function g_1 and a non-increasing function g_2. It follows that for every $\mathbb{R}$-valued function f of bounded variation on a finite interval $[a, b]$ and every $x \in (a, b)$, both one-sided limits $\lim_{t\to x+} f(t)$ and $\lim_{t\to x-} f(t)$ exist. Denote the two limits by $f(x+)$ and $f(x-)$ respectively.

(1.4) Proposition *Let $f(x)$ be a periodic function* of bounded variation *with period $2L$. Let $\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{\frac{\pi\sqrt{-1}\,nx}{L}}$ be the Fourier series attached to $f(x)$, where*

$$\hat{f}(n) := \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{\frac{-\pi\sqrt{-1}\,nt}{L}}\, dt$$

For every $x_0 \in \mathbb{R}$, the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{\frac{\pi\sqrt{-1}\,nx_0}{L}}$$

converges, and

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{\frac{\pi\sqrt{-1}\,nx_0}{L}} = \frac{f(x_0+) + f(x_0-)}{2}\,.$$

(1.5) Proposition *Let f be be a periodic function on $\mathbb{R}$* of bounded variation *with period $2L$. Let $\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{\frac{\pi\sqrt{-1}\,nx}{L}}$ be the Fourier series attached to $f(x)$. Define a periodic function g with period $2L$ on $\mathbb{R}$ by*

$$g(x) = \int_{-L}^{x} (f(t) - \hat{f}(0))\, dt = \int_{-L}^{x} f(t)\, dt - \hat{f}(0)\cdot(x+L), \quad \forall\, x \in \mathbb{R}.$$

Then

$$g(x) = \sum_{0\neq n\in\mathbb{Z}} \hat{f}(n) \int_{-L}^{x} e^{\frac{\pi\sqrt{-1}\,nt}{L}}\, dt = \sum_{0\neq n\in\mathbb{Z}} \frac{\hat{f}(n)}{\pi\sqrt{-1}\,n}\left[e^{\frac{\pi\sqrt{-1}\,nx}{L}} - (-1)^n\right]$$

for all $x \in \mathbb{R}$.

(1.6) Proposition (Parseval identity) *Let $f(x), g(x)$ be* square integrable[1] *periodic functions on $\mathbb{R}$ with period $2L$. Let $\hat{f}(n), \hat{g}(n)$, $n \in \mathbb{Z}$, be the Fourier coefficients of f and g respectively:*

$$\hat{f}(n) = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{\pi\sqrt{-1}\,nx}{L}}\, dx, \quad \hat{g}(n) = \frac{1}{2L}\int_{-L}^{L} g(x)\, e^{\frac{\pi\sqrt{-1}\,nx}{L}}\, dx.$$

(a) $\displaystyle \frac{1}{2L}\int_{-L}^{L} f(x)\,\overline{g(x)}\, dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\,\overline{\hat{g}(n)}$

(b) $\displaystyle \frac{1}{2L}\int_{-L}^{L} |f(x)|^2\, dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$

[1] Square integrable periodic functions on $\mathbb{R}$ include functions of bounded variation and piece-wise continuous functions.

Example. Let f be the periodic function with period 1 such that

$$f(x) = x \quad \forall x \in (-\tfrac{1}{2}, \tfrac{1}{2}).$$

The n-th Fourier coefficient of f is

$$\hat{f}(n) = \int_{-1/2}^{1/2} x\, e^{-2\pi\sqrt{-1}\, nx}\, dx = -\frac{1}{2\pi\sqrt{-1}n} x e^{-2\pi\sqrt{-1}\, nx}\Big|_{x=-1/2}^{x=1/2} = \frac{1}{2\pi\sqrt{-1}n}(-1)^{n+1}.$$

if $n \neq 0$, and

$$\hat{f}(0) = 0.$$

Since

$$\int_{-1/2}^{1/2} x^2\, dx = \frac{1}{12},$$

the equality

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)^2 = \int_{-1/2}^{1/2} f(x)^2\, dx$$

becomes

$$\frac{1}{2\pi^2} \sum_{n=1}^{\infty} n^2 = \frac{1}{12},$$

which simplifies to

$$\sum_{n=1}^{\infty} n^2 = \frac{\pi^2}{6}. \tag{1.6.1}$$

Finally we state a theorem of Fejér on Fourier series of continuous periodic functions.

(1.7) Proposition *Let f be a continuous periodic function on $\mathbb{R}$ with period $2L$. For each positive integer n, denote by S_n the n-th partial sum*

$$S_n(x) = \sum_{|k|\le n} \hat{f}(n)\, e^{\frac{\pi\sqrt{-1}\, nx}{L}} \tag{1.7.1}$$

of the Fourier series of f. Then the arithmetic means

$$\frac{1}{n}\,(S_0 + S_1 + \cdots + S_{n-1})$$

of the partial sums converge uniformly to f. In other words, for every $\epsilon > 0$, there exists a positive integer N_0 such that

$$\left| f(x) - \tfrac{1}{n}\,(S_0(x) + S_1(x) + \cdots + S_{n-1}(x)) \right| < \epsilon, \qquad \forall\, n \ge N_0,\ \forall\, x \in \mathbb{R}.$$

§2. Term-by-term differentiation of Fourier series

We give an account of term-by-term differentiation of Fourier series using a very small part of the theory of generalized function. See [**?**, Ch. 4] for more information about generalized periodic functions and their Fourier series.

(2.1) Generalized functions

The basic phenomenon here is the appearance of certain periodic *generalized functions*, namely Dirac's δ-functions, when we differentiate the generalized function associated to $f(x)$. All periodic functions below have period $2L$, $L > 0$.

A periodic generalized function is not a usual function, and does not necessarily has a definite value at a point. However you can always "integrate" periodic generalized function $\alpha(x)$ against any *smooth* period function $\xi(x)$ on $\mathbb{R}$ and get a well-defined number "$\int_{-L}^{L} \alpha(x)\, \xi(x)\, dx$". The terminology "integrate" is purely formal; no Riemann sum is involved. A generalized function α is a "black box" which for every input smooth periodic function ξ outputs a number; the output is denoted by $\int_{-L}^{L} \alpha(x)\, \xi(x)\, dx$. Smooth periodic function of period $2L$ are often called "test functions" for periodic generalized function of period $2L$.

For every $a \in \mathbb{R}$, we have a periodic generalized function $\delta_{x\equiv a\,(\mathrm{mod} L)}$, which is the mathematical way to represent the idea of a "point source" of unit strength at a point $a \bmod 2L$ of $\mathbb{R}/2L\mathbb{Z}$. It is customary to abuse notation, and write $\delta_{x=a}$ instead of the cluttered notation $\delta_{x\equiv a\,(\mathrm{mod} L)}$ for this Dirac's δ-function. It is defined by

$$\int_{-L}^{L} \delta_{x=a}(x)\, \xi(x)\, dx = \xi(a) \tag{2.1.1}$$

for every $a \in \mathbb{R}$ and every smooth periodic function $\xi(a)$ with period $2L$.

Every piece-wise smooth periodic function $g(x)$ with period $2L$ gives rise to a generalized function $[g]$, defined by

$$\int_{-L}^{L} [g](x), \xi(x)\, dx := \int_{-L}^{L} g(x)\, \xi(x)\, dx \tag{2.1.2}$$

for every smooth periodic $\xi(a)$ with period $2L$. Therefore the derivative $[g]'$ of $[g]$ is defined by

$$\int_{-L}^{L} [g]'(x)\, \xi(x)\, dx = -\int_{-L}^{L} g(x)\, \xi'(x)\, dx \tag{2.1.3}$$

(2.2) Notation. In this subsection $f(x)$ denotes a piece-wise smooth periodic function on $\mathbb{R}$ with period $2L$. Let $x_1, \ldots, x_m$ be non-smooth points of $f(x)$, $-L < x_1 < \cdots < x_m \le L$, such that for every non-smooth points differs from exactly one of the x_i's by an integer multiple of $2L$.

Let $j_1, \ldots, j_m$ be the jumps of $f(x)$ at the non-smooth points, and let $j'_1, \ldots, j'_m$ be the jumps of the derivative $f'(x)$ of $f(x)$ at the non-smooth points. In other words

$$\begin{aligned} j_k &:= f(x_k+) - f(x_k-) = \lim_{x \to x_k+} f(x) - \lim_{x \to x_k-} f(x) \\ j'_k &:= f'(x_k+) - f'(x_k-) = \lim_{x \to x_k+} f'(x) - \lim_{x \to k_i-} f'(x) \end{aligned} \tag{2.2.1}$$

for $k = 1, \ldots, m$. Let

$$f(x) \sim \sum_{n \in \mathbb{Z}} c_n \, e^{\frac{\pi\sqrt{-1}nx}{L}},$$

i.e. the above infinite series is the Fourier series attached to the given piece-wise smooth periodic function $f(x)$ with period $2L$, where

$$c_n := \frac{1}{2L} \int_{-L}^{L} f(x) \, e^{\pi\sqrt{-1}\,nx} L \, dx, \quad \forall \, n \in \mathbb{Z}.$$

Similarly let

$$f'(x) \sim \sum_{n \in \mathbb{Z}} c'_n \, e^{\frac{\pi\sqrt{-1}nx}{L}}$$

and

$$f''(x) \sim \sum_{n \in \mathbb{Z}} c''_n \, e^{\frac{\pi\sqrt{-1}nx}{L}}$$

be the Fourier series attached to the periodic piece-wise continuous functions $f'(x)$ and $f''(x)$.

(2.2.2) Proposition *The derivative of the generalized function $[f]$ is*

$$[f]' = [f'] + \sum_{k=1}^{m} j_k \cdot \delta_{x=x_k} \, . \tag{2.2.3}$$

(2.2.4) Corollary *The second derivative of the generalized function $[f]$ is*

$$[f]'' = [f''] + \sum_{k=1}^{m} j'_k \cdot \delta_{x=x_k} + \sum_{k=1}^{m} j_m \cdot \delta'_{x=x_k}. \tag{2.2.5}$$

for every periodic test function $\xi(x)$.

Integrating both sides of (2.2.3) and (2.2.5) against $e^{\frac{-\pi\sqrt{-1}\,nx}{L}}$, we get corollary 2.2.6 below. 2.2.4, by

(2.2.6) Corollary *Let f be a periodic piece-wise smooth function of period $2L$. Let $j_1,\ldots,j_m$ and $j'_1,\ldots,j'_m$ be the jumps at the singular points $x_1,\ldots,x_m$ of f and f' respectively. Let*

$$\sum_{n\in\mathbb{Z}} c_n e^{\frac{\pi\sqrt{-1}nx}{L}}, \quad \sum_{n\in\mathbb{Z}} c'_n e^{\frac{\pi\sqrt{-1}nx}{L}}, \quad \sum_{n\in\mathbb{Z}} c''_n e^{\frac{\pi\sqrt{-1}nx}{L}}$$

be the Fourier series of f, f', f'' respectively. Then

$$\tfrac{\pi\sqrt{-1}n}{L}\, c_n = c'_n + \sum_{k=1}^{m} \tfrac{1}{2L}\, j_k\, e^{\frac{-\pi\sqrt{-1}nx_k}{L}}, \tag{2.2.7}$$

and

$$\tfrac{-\pi^2 n^2}{L^2}\, c_n = c'_n + \sum_{k=1}^{m} \tfrac{1}{2L}\, j'_k\, e^{\frac{-\pi\sqrt{-1}nx_k}{L}} + \sum_{k=1}^{m} \tfrac{1}{2L} j_k \tfrac{\pi\sqrt{-1}n}{L}\, e^{\frac{-\pi\sqrt{-1}nx_k}{L}}. \tag{2.2.8}$$

(2.3) The Fourier series of a periodic generalized function

For every periodic generalized function α of period $2L$, define a function $\hat{\alpha}:\mathbb{Z}\to\mathbb{C}$ by

$$\hat{\alpha}(n) := \frac{1}{2L}\int_{-L}^{L} \alpha(x)\, e^{\frac{-\pi\sqrt{-1}nx}{L}}\, dx\,, \tag{2.3.1}$$

and the series

$$\sum_{n\in\mathbb{Z}} \hat{\alpha}(n)\, e^{\frac{\pi\sqrt{-1}nx}{L}} \tag{2.3.2}$$

is called the Fourier series of α.

The following lemma says that the Fourier series of the derivative α' of α is the term-by-term derivative of the Fourier series of α, for every periodic generalized function α on $\mathbb{R}$. This is conceptually very satisfactory, especially if compared with §3.4 of Haberman.

(2.3.3) Lemma *The Fourier series attached to the derivative α' of a generalized periodic function α of period $2L$ is*

$$\sum_{n\in\mathbb{Z}} \hat{\alpha}(n)\, \tfrac{\pi\sqrt{-1}n}{L} e^{\frac{\pi\sqrt{-1}nx}{L}},$$

which is the term-by-term derivative of the Fourier series of α of α.

Proof.

$$\begin{aligned}\hat{\alpha'}(n) = \tfrac{1}{2L}\int_{-L}^{L} \alpha'(x)\, e^{-\frac{\pi\sqrt{-1}\,nx}{L}}\, dx &= -\tfrac{1}{2L}\int_{-L}^{L} \alpha(x)\, \frac{d}{dx} e^{-\frac{\pi\sqrt{-1}\,nx}{L}}\, dx \\ &= \tfrac{\pi\sqrt{-1}n}{L}\tfrac{1}{2L}\int_{-L}^{L} \alpha(x)\, e^{-\frac{\pi\sqrt{-1}\,nx}{L}}\, dx \;= \tfrac{\pi\sqrt{-1}n}{L}\tfrac{1}{2L}\,\hat{\alpha}(n).\end{aligned}$$

(2.3.4) Proposition *Let α be a periodic generalized function of period $2L$. The Fourier series*

$$\sum_{n\in\mathbb{Z}} \hat{\alpha}(n)\, e^{\frac{\pi\sqrt{-1}nx}{L}}$$

of α converges to α in the spaces of generalized function in the following sense: for every smooth periodic function ξ of period $2L$, the series

$$\sum_{n\in\mathbb{Z}} \hat{\alpha}(n) \int_{-L}^{L} e^{\frac{\pi\sqrt{-1}nx}{L}}\, \xi(x)\, dx$$

converges to

$$\int_{-L}^{L} \alpha(x)\, \xi(x)\, dx.$$