LETTERS TO THE EDITOR.

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Fourier's Series.

IN reply to Mr. Love's remarks in NATURE of October 13, I would say that in the series

$$y = \sin x + \tfrac{1}{2}\sin 2x + \ . \ . \ . \ + \frac{1}{n-1}\sin(n-1)x + \frac{1}{n}\sin nx,$$

in which $\frac{1}{n}\sin nx$ is the last term considered, x must be taken smaller than π/n in order to find the values of y in the immediate vicinity of $x = 0$.

If it is inadmissible to stop at "any convenient nth term," it is quite as illogical to stop at the equally "convenient" value π/n.

ALBERT A. MICHELSON.

The University of Chicago Ryerson Physical Laboratory, Chicago, December 1.

I SHOULD like to add a few words concerning the subject of Prof. Michelson's letter in NATURE of October 6. In the only reply which I have seen (NATURE, October 13), the point of view of Prof. Michelson is hardly considered.

Let us write $f_n(x)$ for the sum of the first n terms of the series

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \&c.$$

I suppose that there is no question concerning the form of the curve defined by any equation of the form

$$y = 2f_n(x).$$

Let us call such a curve C_n. As n increases without limit, the curve approaches a limiting form, which may be thus described. Let a point move from the origin in a straight line at an angle of 45° with the axis of X to the point (π, π), thence vertically in a straight line to the point $(\pi, -\pi)$, thence obliquely in a straight line to the point $(3\pi, \pi)$, &c. The broken line thus described (continued indefinitely forwards and backwards) is the limiting form of the curve as the number of terms increases indefinitely. That is, if any small distance d be first specified, a number n' may be then specified, such that for every value of n greater than n', the distance of any point in C_n from the broken line, and of any point in the broken line from C_n, will be less than the specified distance d.

But this limiting line is not the same as that expressed by the equation

$$y = \lim_{n=\infty} 2f_n(x).$$

The vertical portions of the broken line described above are wanting in the locus expressed by this equation, except the points in which they intersect the axis of X. The process indicated in the last equation is virtually to consider the intersections of C_n with fixed vertical transversals, and seek the limiting positions when n is increased without limit. It is not surprising that this process does not give the vertical portions of the limiting curve. If we should consider the intersections of C_n with horizontal transversals, and seek the limits which they approach when n is increased indefinitely, we should obtain the vertical portions of the limiting curve as well as the oblique portions.

It should be observed that if we take the equation

$$y = 2f_n(x),$$

and proceed to the limit for $n = \infty$, we do not necessarily get $y = 0$ for $x = \pi$. We may get that ratio by first setting $x = \pi$, and then passing to the limit. We may also get $y = 1$, $x = \pi$, by first setting $y = 1$, and then passing to the limit. Now the limit represented by the equation of the broken line described above is not a special or partial limit relating solely to some special method of passing to the limit, but it is the complete limit embracing all sets of values of x and y which can be obtained by any process of passing to the limit.

J. WILLARD GIBBS.

New Haven, Conn., November 29.

FOURIER'S series arises in the attempt to express, by an infinite series of sines (and cosines) of multiples of x, a function of x which has given values in an interval, say from $x = -\pi$ to $x = \pi$. There is no "curve" in the problem. Curves occur in the solution of the problem, and there they occur by way of illustration. There are two sorts of curves which occur. In the first place, taking $\phi(x)$ as the function to be expressed by the series, and $f(x)$ as the sum of the series, we have the curves $y = \phi(x)$ and $y = f(x)$, the graphs of the two functions. These coincide wherever the series expresses the function; but, if the function $\phi(x)$ is one which cannot be expressed by a Fourier's series for all values of x in the interval, the curves do not coincide throughout the interval. In the second place, taking $f_n(x)$ as the sum of the first n terms of the series, we have the family of curves $y = f_n(x)$, the graphs of $f_n(x)$ for different values of n. As n increases the graphs of $f(x)$ and $f_n(x)$ approach to coincidence in the sense that, if any *particular value* of x is taken, and any small distance d is specified, a number n' may then be specified such that for every n greater than n', the difference of the ordinates of the two curves is less than d. But this is not the same thing as saying that the curves tend to coincide geometrically, and they do not in fact lie near each other in the neighbourhood of a finite discontinuity of $\phi(x)$. It is usual to illustrate the tendency to discontinuity of $f(x)$ by noting the form of the curve $y = f_n(x)$ for large values of n, but the shape of this curve always fails to give an indication of the sum of the series for the particular values of x for which $\phi(x)$ and $f(x)$ are discontinuous. This is the case in the example cited by Prof. Willard Gibbs, where all particular values between $-\pi$ and π are equally indicated by the curve $y = f_n(x)$, but the sum of the series is precisely zero.

May I point out that there is some ambiguity in the expression "the limiting form of the curve" used by Prof. Willard Gibbs? Taking his example, it is quite true that n' can be taken so great that, for every n greater than n', there is a point of C_n within the given distance d of any point on the broken line, but this statement is not quite complete. It is also true that a number n can be taken great enough to bring the point of C_n on *any assigned* ordinate within the given distance d of its *ultimate position* on the broken line, but it is further essential to observe that no number n can be taken great enough to bring *every* point of C_n within the given distance d of its ultimate position on the broken line. The number n which succeeds for any one ordinate always fails for some other ordinate. Suppose, to fix ideas, that we take a point on C_n for which $y = 1$, and x is nearly π, so that $\pi - x$ is less than d, and keeping x fixed, observe how y changes when n increases; it will be found that, for values of m very much greater than n, the ordinate of C_m, for this x is very nearly π, and we can in fact take m great enough to make this ordinate lie between π and $\pi - d$. In words, the representative point, which begins by nearly coinciding with a point on a vertical part of the broken line, creeps along the line, and ends by coinciding with a point on the oblique part of the broken line. This will be the case for every value of x, near $x = \pi$, with the single exception of the value π. Thus, in the passage to the limit, every point near the vertical part of the broken line disappears from the graph, except the points on the axis of x. This peculiarity is always presented by a series whose sum is discontinuous; in the neighbourhood of the discontinuity the series does not converge uniformly, or the graph of the sum of the first n terms is always appreciably different from the graph of the limit of the sum.

In this way the graph of the sum of the first n terms fails to indicate the behaviour of the function expressed by the limit of this sum, and we may illustrate the distinction between the two, as Prof. Willard Gibbs does, by considering the intersections of the graph with lines parallel to the axis of x. Keeping y fixed, say $y = 1$, we may find, in his example, a number n, so that there is a corresponding value of x differing from π by less than d, and then, allowing n to increase indefinitely, we shall get a series of values of x, having π as limiting value. But this limiting value is *not attained*. In Prof. Willard Gibbs's notation, the equation $2f_n(x) = 1$ has a root near to π when n is great, and n can be taken so great that the root differs from π by less than any assigned fraction; but the equation

$$\lim_{n=\infty} 2f_n(x) = 1$$

has no real root. In fact Prof. Willard Gibbs's "limiting form of the curve" corresponds to limits which are not attained; but the limiting form in which the vertical portions of the broken line are replaced by the points where they cut the axis of x corresponds to limits which are effectively attained. It is the