MATH 314 PRACTICE PROBLEMS, MARCH 2016

1. Give an example of a linear operator J on a non-zero finite dimensional vector space V over $\mathbb{R}$ such that $J^2 + \mathrm{Id}_V = 0$, the zero operator.

2. Give an example of a non-zero linear operator N on a vector space V over $\mathbb{R}$ such that $N^2 = 0$.

3. Let U_1, U_2, U_3 be vector subspaces of a finite-dimensional vector space V over a field F. Prove that

$$\dim_F(U_1 \cap U_2 \cap U_3) \geq \sum_{i=1}^{3} \dim_F(U_i) - 2\dim_F(V).$$

4. Let U, M, N be vector subspaces of a vector space V over a field F.

(a) Show that $U \cap (M+N) \supseteq (U \cap M) + (U \cap N)$.

(b) Prove that if $U \supseteq M$, then $U \cap (M+N) = M + (U \cap N)$.

5. Let V be the set of all smooth $\mathbb{C}$-valued functions $f(t)$ on $\mathbb{R}$ such that

$$\frac{d^3 f}{dt^3} + 3\frac{d^2 f}{dt^2} + 3\frac{df}{dt} + f = 0$$

(a) Let $\mathscr{F}$ be the $\mathbb{C}$-vector space consisting of all $\mathbb{C}$-valued functions on $\mathbb{R}$. Show that V is a $\mathbb{C}$-vector subspace of $\mathscr{F}$.

(b) Show that $\frac{d}{dt}$ induces a linear operator T on V.

(c) Find a basis of V.

(d) Determine the matrix representation of the linear operator T with respect to the basis you gave in (c) above.

6. Notation as in problem 5 above. Let $\lambda : V \to \mathbb{C}$ be the function on V which sends every element $f(t) \in V$ to $f(0)$, the value of the function $f(t)$ at $t = 0$.

(a) Show that λ is an element of the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of V.

(b) Let T^t be the transpose of the linear operator T on V. Compute the value of $T^t(\lambda)$ at the element $e^{-t} \in V$.

(c) Compute $\mathrm{Ker}(T^t)$, the kernel of T^t, and find a $\mathbb{C}$-basis of $\mathrm{Ker}(T^t)$.

7. Let T be an F-linear operator on a vector space V over a field F. Suppose that $f(x) \in F[x]$ is a polynomial such that $f(1) = 0$, $f'(1) \neq 0$, $f(T) = 0 \cdot \mathrm{Id}_V$, and the ideal of $F[x]$ consisting of all polynomials $g(x) \in F[x]$ such that $g(T) = 0 \cdot \mathrm{Id}_V$ is generated by $f(x)$. Show that there exists a non-zero vector $v \in V$ such that $T(v) = v$.

8. Let T be a linear operator on a real vector space V such that $T^3 = \mathrm{Id}_V$. Let $U := \mathrm{Im}(T - \mathrm{Id})$ and let $W := \mathrm{Im}(T^2 + T + \mathrm{Id}_V)$.

(a) Show that $U \cap W = (0)$.

(b) Show that $U + W = V$.

(Hint: The assumption means that $f(T) = 0$ for the polynomial $f(x) = x^3 - 1 = (x-1)(x^2+x+1)$.