MATH 314 PRACTICE PROBLEMS, APRIL 2016

1. True/False question. Give a proof if true, and give a counter-example if false.

- Every square matrix with entries in $\mathbb{R}$ is conjugate (by an invertible matrix with entries in $\mathbb{R}$) to an upper-triangular matrix.
- If the minimal polynomial of linear operator T on a finite dimensional vector space over $\mathbb{C}$ is x^3, then the dimension of the image of T is at least 2.
- Suppose that the characteristic polynomial of a square matrix with entries in $\mathbb{C}$ is $(x-2)^3(x+7)$, then this matrix is not diagonalizable.
- Let T be a linear operator on a finite dimensional vector space over a field F. Let $f(x)$ be a polynomial in $F[x]$ such that $f(x)$ is relatively prime to the minimal polynomial of T. Then $f(T)$ is an isomorphism.

2. Let T be a linear operator on a finite dimensional vector space over $\mathbb{Q}$. Suppose that the characteristic polynomial is $(x^3-1)^3$. What are the possible rational canonical forms of T?

3. Let P_4 be the 5-dimensional vector space over $\mathbb{R}$ consisting of all polynomials in one variable x of degree at most 4, with inner product defined by

$$(f|g) := \int_{-1}^{1} f(x)\,g(x)\,dx.$$

Let $D \in \mathrm{End}_{\mathbb{R}}(P_4)$ be the linear operator on P_4 induced by $\frac{d}{dx}$. Let D^* be the adjoint of D. Find $D^*(x^4+x^3+x^2+x+1)$.

4. Let P_4 be the 5-dimensional vector space over $\mathbb{R}$ consisting of all polynomials in one variable x of degree at most 4, and let $D \in \mathrm{End}_{\mathbb{R}}(P_4)$ be the linear operator on P_4 induced by $\frac{d}{dx}$. Find the rational caonical form for the linear operator D^2+D+1.

5. Give an example of a unitary operator U on a 5-dimensional vector space V such that $\mathrm{Ker}(U-\mathrm{Id}_V)$ is 3-dimensional.

6. Given an example of a linear operator on a 5-dimensional vector space V with an inner product which is *not* normal.

7. Given an example of two linear operator S_1, S_2 on a finite dimensional vector space V such that both S_1 and S_2 are diagonalizable but $\{S_1, S_2\}$ is not simultaneously diagonalizable.

8. Let $\mathcal{F}$ be the family consisting of all 2×2 matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Is $\mathcal{F}$ simultaneously diagonalizable over $\mathbb{C}$?