Math 315 practice problems, December 4, 2015

1. (True/False questions with proofs, 30 points) Decide whether of the following statements are true or false. Provide a proof if it is true; give a counter-example if it is false.

(a) Let $n \geq 2$ be positive integers. If $a^n \equiv a \pmod{n}$ for all integers $a$, then $n$ is a prime number.

(b) Let $n \geq 2$ be a positive integer, and let $a$ be an integer. If the congruence equation $x^3 \equiv a \pmod{n}$ has an integer solution, then the congruence equation $x^3 \equiv a \pmod{n^2}$ has an integer solution.

(c) If $p$ is an odd prime number such that $p+1 \equiv 0 \pmod{8}$, then there exists a integer $a$ such that $a^2 \equiv 2 \pmod{p^3}$.

(d) Let $f(X)$ be a monic polynomial with integer coefficients. Let $p$ be an odd prime number. Suppose that for every positive integer $m$, there exists an integer $a$ such that $f(a) \equiv 0 \pmod{p^m}$. Then there exists an integer $b$ such that $f(b) = 0$.

(e) Let $a$ be an integer, and let $n_1, n_2$ be odd positive integers. If $n_1 \equiv n_2 \pmod{a}$, then $\left(\frac{a}{n_1}\right) = \left(\frac{a}{n_2}\right)$.

2. Let $n$ be a positive integer. Give a necessary and sufficient condition for the congruence equation

$$x^2 \equiv 1 \pmod{n}$$

to have exactly 2 solutions in $\mathbb{Z}/n\mathbb{Z}$. Please provide a complete proof.