Math 350 Practice Problems
April, 2005

1. Prove that \[ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \] for all \( n \in \mathbb{N}_{>0}. \)

2. Prove that \[ \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right)^n = \left( \begin{array}{cc} 1 & 2n \\ 0 & 1 \end{array} \right) \] for all \( n \in \mathbb{N}_{>0}. \)

3. Determine whether the following statements are true or false.
(a) For prime numbers \( p, \) the Legendre symbol \( \left( \frac{5}{p} \right) \) depends only on the congruence class of \( p \) modulo 5.
(b) For prime numbers \( p, \) the Legendre symbol \( \left( \frac{11}{p} \right) \) depends only on the congruence class of \( p \) modulo 11.
(c) For non-zero natural numbers \( a, b \) which are relatively prime, the Jacobi symbol \( \left( \frac{a}{b} \right) \) depends only on the congruence class of \( a \mod b. \)
(d) For non-zero natural numbers \( a, b \) which are relatively prime, the Jacobi symbol \( \left( \frac{b}{a} \right) \) depends only on the congruence class of \( b \mod 4a. \)

4. Find all integers \( n \) such that \(-1000 \leq n \leq 1000\) and satisfying the following three congruence relations
\[ n \equiv 2 \pmod{3}, \quad n \equiv 3 \pmod{5} \quad \text{and} \quad n \equiv 4 \pmod{7}. \]

5. For \( p = 173 \) and \( p = 401, \) determine the set of all elements \( x \in \mathbb{Z}/p^2\mathbb{Z} \) such that \( x^5 \equiv 1 \pmod{p^2}. \)

6. Determine the set of all \( x \in \mathbb{Z}/13^4\mathbb{Z} \) such that \( x^3 \equiv -1 \pmod{13^4}. \)

7. Let \( S \) be the set of all pairs \((a, b)\) with \( a, b \in \mathbb{Z}, \) \( 0 \leq a, b \leq 20 \) such that there exists an integer \( x \) such that \( x \equiv a \pmod{36} \) and \( x \equiv b \pmod{100}. \) Determine the number of elements of \( S. \)

8. Let \( p, q \) be prime numbers, \( p \neq q. \) Find a natural number \( n \) with \( 0 \neq n < pq \) such that \( p^{2q-1} + q^{2p-1} \equiv n \pmod{pq}. \) (The number \( n \) should be given in terms of \( p \) and \( q. \))

9. Let \( p \) be an odd prime number. Show that the Legendre symbol \( \left( \frac{7}{p} \right) \) depends only on the congruence class of \( p \mod 28, \) and determine the value of \( \left( \frac{7}{p} \right) \) for each congruence class of \( p \mod 28. \)
10. (a) Determine the simple continued fraction expansion of $\sqrt{7}$. 
(b) Find natural numbers $a, b, c, d$ such that $\frac{c}{d} < \frac{\sqrt{7}}{2} < \frac{a}{b}$, $b, d > 100$, and $ad - bc = 1$.

11. Does the quadratic congruence equation

$$x^2 + 2x + 1002 \equiv 0 \pmod{483}$$

have a solution in $\mathbb{Z}/483\mathbb{Z}$?

12. Expand $\frac{173}{409}$ as a simple continued fraction.

13. Find natural numbers $a, b$ such that $a 409 - b 250 = 1$.

14. Let $p$ be a prime number. Determine the following numbers in terms of $p$.
   (a) the number of quadratic non-residues modulo $p$,
   (b) the number of primitive elements in $(\mathbb{Z}/p\mathbb{Z})^\times$,
   (c) the number of non-primitive elements in $(\mathbb{Z}/p\mathbb{Z})^\times$,
   (d) the number of elements in $(\mathbb{Z}/p\mathbb{Z})^\times$ which are quadratic non-residues but not primitive.

15. Determine the number of elements of $(\mathbb{Z}/9797\mathbb{Z})^\times$ of order 100.

16. (a) What is the maximal possible order for elements of $(\mathbb{Z}/9797\mathbb{Z})^\times$?
   (b) Determine the number of elements of $(\mathbb{Z}/9797\mathbb{Z})^\times$ whose order are maximal possible.

17. Prove that 561 is an Euler pseudoprime to the base 2, i.e.

$$2^{280} \equiv \left( \frac{2}{561} \right) \pmod{561},$$

where $\left( \frac{2}{561} \right)$ is the Jacobi symbol.

18. Suppose that $n$ is natural number, $n \equiv 5 \pmod{12}$ and that $n$ is an Euler pseudoprime to the base 3. Prove that $n$ is a strong pseudoprime to the base 3, i.e. $n$ passes the Miller-Rabin test to the base 3.

19. Relate the length of the period of the decimal expansion of $\frac{1}{561}$ to the order of a suitable element in $(\mathbb{Z}/n\mathbb{Z})^\times$ for a suitable integer $n$, and determine the length of that period.
20. The number 1729 factors as $1729 = 7 \times 13 \times 19$.

(a) Determine the number of elements in $(\mathbb{Z}/1729)^\times$ of order 3.

(b) Determine the number of elements in $(\mathbb{Z}/1729)^\times$ which are squares, i.e. equal to the square of some element in $(\mathbb{Z}/1729)^\times$.

(c) Determine the number of elements in $(\mathbb{Z}/1729)^\times$ which are cubes, i.e. equal to the cube of some element in $(\mathbb{Z}/1729)^\times$.

(c) Determine the number of elements in $(\mathbb{Z}/1729)^\times$ which are fourth powers, i.e. congruent to $x^4$ modulo 1729 for some integer $x$. 