Math 350 practice problems, April 15, 2015

The second in-class exam focuses on the basics, just as the first one. Some of the problems below are more difficult than the exam problems.

1. (True/False questions with proofs, 30 points) Decide whether the following statements are true or false. Provide a proof if it is true; give a counter-example if it is false.

(a) Let $n \geq 2$ be positive integers. If $a^n \equiv a \pmod{n}$ for all integers a, then n is a prime number.

(b) Let $n \geq 2$ be a positive integer, and let a be an integer. If the congruence equation $x^3 \equiv a \pmod{n}$ has an integer solution, then the congruence equation $x^3 \equiv a \pmod{n^2}$ has an integer solution.

(c) If p is an odd prime number such that $p+1 \equiv 0 \pmod{8}$, then there exists a integer a such that $a^2 \equiv 2 \pmod{p^3}$.

(d) Let $f(X)$ be a monic polynomial with integer coefficients. Let p be an odd prime number. Suppose that for every positive integer m, there exists an integer a such that $f(a) \equiv 0 \pmod{p^m}$. Then there exists an integer b such that $f(b) = 0$.

(e) Let a be an integer, and let n_1, n_2 be odd positive integers. If $n_1 \equiv n_2 \pmod{a}$, then $\left(\frac{a}{n_1}\right) = \left(\frac{a}{n_2}\right)$.

(f) Let n, a be positive integers. If $a \not\equiv 0 \pmod{n}$, then $a^2 \not\equiv 0 \pmod{n^2}$.

(g) Suppose that u, v are integers such that $u^2 + 3v^2$ is equal to a prime number p. Then $p \equiv 1 \pmod{3}$.

2. Let n be a positive integer. Give a necessary and sufficient condition for the congruence equation

$$x^2 \equiv 1 \pmod{n}$$

to have exactly 2 solutions in $\mathbb{Z}/n\mathbb{Z}$. Please provide a complete proof.

3. Determine the number of elements in $\mathbb{Z}/13^5\mathbb{Z}$ (respectively $\mathbb{Z}/(11 \times 13^2)\mathbb{Z}$) which are cubes, i.e. the third power of elements of $\mathbb{Z}/13^5\mathbb{Z}$ (respectively $\mathbb{Z}/(11 \times 13^2)\mathbb{Z}$).

4. (Extra credit) Give a necessary and Give a necessary and sufficient condition for the congruence equation

$$x^3 \equiv 1 \pmod{n}$$

to have exactly 3 solutions in $\mathbb{Z}/n\mathbb{Z}$. Please provide a complete proof.