MATH 350 ASSIGNMENT 3, SPRING 2017

Due in class on Monday, February 6

Part 1. From the textbook *A friendly introduction to number theory*.

- Exercise 11.12 (c) Note. You need to give a list of values of n for which $\phi(n) = n/12$, and show that there is no other positive integers for which this equality holds.
- Exercise 12.4 (a), and (the first part, if you use the 3rd edition) of (b). For part (b), please formulate the pattern you find for $A_m \pmod{m^2}$ as a precise statement, a conjecture.

  Note: Exercise 12.4 in the 3rd edition has been changed in the 4th edition: The second part of Exercise 12.4 (b) was removed from part (b) and became part (c) in the 4th edition.
- Exercise 12.6

Part 2.

A. Define a function $f : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$ by

$$f(n) = \sum_{d|n} \phi(d)$$

(a) Show that the function f is *multiplicative*, i.e.

$$f(mn) = f(m) \cdot f(n) \qquad \text{if } \gcd(m,n) = 1.$$

(b) Show that $f(n) = n$ for every positive integer n. In other words

$$\sum_{d|n} \phi(d) = n \quad \forall n \in \mathbb{N}_{\geq 1}.$$

B. Define a function from $\mathbb{R}$ to $[0,1)$ by

$$x \mapsto \langle x \rangle := x - \lfloor x \rfloor.$$

The number $\langle x \rangle$ is called the *fractional part* of x. Suppose that $n \geq 2$ is a positive integer, a, b are integers with $\gcd(a,n) = 1$. Prove that

$$\sum_{0 \leq k < n} \left\langle \tfrac{ak+b}{n} \right\rangle = (n-1)/2.$$

Part 3. Extra credit problems.

C. Prove the conjecture you formulated in part (b) of exercise 12.4. (This is Exercise 12.4 (c) in the 4th edition.)

D. Let p be a prime number such that $p \equiv 1 \pmod 3$. Let n be any integer. Show that there exists an integer k such that $n \equiv k^3 \pmod p$ if and only if $n^{(p-1)/3} \equiv 1 \pmod 3$.