MATH 350 ASSIGNMENT 4, SPRING 2017

Due in class on Monday, February 13

Part 1. From the textbook *A friendly introduction to number theory*.

- Exercise 12.5 (a)–(f).
- Exercise 28.14 (a), (b), 4th edition (= Exercise 21.14 (a), (b), 3rd edition)
- Exercise 28.5 (a), (b), (c), 4th edition) (= Exercise 21.6 (a), (b), (c), 3rd edition)

Part 2.

A. The *Möbius function* $\mu(n)$ is the function from $\mathbb{N}_{\geq 1}$ to $\{1,-1\}$, defined as follows. Suppose that

$$n = p_1^{e_1} \cdots p_f^{e_f}, \ \ f \in \mathbb{N}, \ e_1, \ldots, e_f \in \mathbb{N}_{\geq 1}$$

is the factorization of n, where $p_1, \ldots, p_f$ are mutually distinct prime numbers. Then

$$\mu(n) = \begin{cases} 0 & \text{if } \ e_i \geq 2 \text{ for some } i = 1, \ldots, f \\ (-1)^f & \text{if } \ e_i = 1 \text{ for all } i = 1, \ldots, f. \end{cases}$$

Prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{if } \ n \geq 2 \end{cases}$$

B. Let $f(n)$ be the maximum of the order of elements in $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

(a) Determine the value of $f(n)$ for $n = 40, 200, 207, 425$.

(b) Formulate a conjecture/reasonable guess for the value of $f(n)$ for general n's.

Part 3. Extra credit problems.

C. Exercise 28.14 (c), 4th edition (= Exercise 21.14 (c), 3rd edition)

D. Prove the conjecture you formulated in part (b) of problem B above.

E. Let $n \geq 2$ be a positive integer. Let a be a non-zero integer with $\gcd(a,n) = 1$. Show that

$$\sum_{1 \leq k \leq n-1, \, \gcd(k,n)=1} \left\langle \tfrac{ak}{n} \right\rangle = \phi(n)/2,$$

where $\left\langle \frac{ak}{n} \right\rangle = \frac{ak}{n} - \lfloor \frac{ak}{n} \rfloor$ is the fractional part of $\frac{ak}{n}$ defined in assignment 3.

F. Let $n \geq 2$ be a positive integer. Prove that

$$\sum_{1 \leq k \leq n-1, \, \gcd(k,n)=1} e^{2\pi\sqrt{-1}\,k/n} = \mu(n),$$

where $\mu(n)$ is the Möbius function of n.