Math 371 Review 1

Some Notation and Basic Concepts

Linear Algebra

- fields.
- Examples of fields:
  - the field of rational, real and complex numbers: \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \)
  - \( \mathbb{F}_p := \) the finite field with \( p \) elements, where \( p \) is a prime number
  - the field of rational functions, \( \mathbb{Q}(x), \mathbb{R}(x), \mathbb{C}(x) \), consisting of quotients of polynomials with coefficients in \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \) respectively.
- vector spaces, subspaces.
- dimension, basis, linear independence.
- linear transformations (also called “homomorphisms between vector spaces”)
- matrix representations of a linear transformation. (So, after choosing a basis of the source and a basis of a target vector space, a linear transformation is represented by a matrix.) Effect of change of bases on the matrix representation.
- direct sum of vector spaces
- eigenvalues, eigenvectors, and eigenspaces (of a linear transformation), characteristic polynomials.
- diagonalization (of a linear transformation)

Basic Group Theory

- groups, subgroups
- Examples of groups
  - cyclic groups: \( \mathbb{Z} := \) integers, \( \mathbb{Z}/n\mathbb{Z} = \) a cyclic group with \( n \) elements.
  - the dihedral group \( D_{2n} \) with \( 2n \) elements; it is the group of all symmetries of a regular \( n \)-gon.
  - the quaternion group \( \mathbb{Q}_8 \) with 8 elements.
  - the “general linear groups” \( \text{GL}_n(\mathbb{Q}), \text{GL}_n(\mathbb{R}), \text{GL}_n(\mathbb{C}), \text{GL}_n(\mathbb{F}_p) \). These are the group of invertible \( n \times n \) matrices with coefficients in \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \), and \( \mathbb{F}_p \) respectively. Notice that \( \text{GL}_1(\mathbb{Q}) = \mathbb{Q}^\times \), the group of non-zero rational numbers under multiplication; similarly \( \text{GL}_1(\mathbb{R}) = \mathbb{R}^\times \), \( \text{GL}_1(\mathbb{C}) = \mathbb{C}^\times \), \( \text{GL}_1(\mathbb{F}_p) = \mathbb{F}_p^\times \).
the “special linear groups” $\text{SL}_n(\mathbb{Q})$, $\text{SL}_n(\mathbb{R})$, $\text{SL}_n(\mathbb{C})$ and $\text{SL}_n(\mathbb{F}_p)$; each is the subgroup of the respective general linear group consisting of all elements whose determinant is equal to 1.

the “projective linear groups”: $\text{PGL}_n(\mathbb{Q}) := \text{GL}_n(\mathbb{Q})/\mathbb{Q}^\times \cdot I_n$, the other groups $\text{PGL}_n(\mathbb{R})$, $\text{PGL}_n(\mathbb{C})$, and $\text{PGL}_n(\mathbb{F}_p)$ are defined similarly.

the symmetric group $S_n$ with $n!$ elements, consisting of all permutations of the set $\{1, 2, \ldots, n\}$.

the alternating group $A_n$, equal to the subgroup of index 2 in $A_n$ consisting of all even permutations in $S_n$.

• cosets

• homomorphisms of groups

• $\text{Ker}(h) :=$ the kernel of a homomorphism $h : G \rightarrow G'$ between groups, $\text{Image}(h) :=$ the image of a homomorphism $h : G \rightarrow G'$ between groups.

• normal subgroups, quotient groups

• the product of (a finite number of) groups

• conjugacy class (of an element in a group)

• $\text{Z}_G(x) :=$ the centralizer of an element $x$ in a group $G$, $\text{Z}_G(H) :=$ the centralizer of a subgroup $H$ of a group $G$.

• $\text{N}_G(x) :=$ the normalizer of an element $x$ in a group $G$, $\text{N}_G(H) :=$ the normalizer of a subgroup $H$ in a group $G$.

Review Problems

1. Let $V$ be the $\mathbb{R}$-vector space consisting of all polynomials of degree at most 3 with coefficients in $\mathbb{R}$. Let $D : V \rightarrow V$ be the linear transformation which sends every polynomial $f(x)$ in $V$ to the derivative $\frac{df}{dx}$. Let $\partial : V \rightarrow V$ be the linear transformation which sends every polynomial $f(x)$ in $V$ to $x \frac{df}{dx}$.

   (i) Show that $\{1, x, x^2, x^3\}$ form a basis of $V$.

   (ii) Find the matrix representation of $D$ and $\partial$ with respect to the basis in (i) above.

   (iii) Let $n$ be a positive integer. Find a basis of $\text{Ker}(D^n)$ and a basis of $\text{Image}(D^n)$.

   (iv) Find $\text{Ker}(\partial^n)$ and $\text{Image}(\partial^n)$ for a positive integer $n$.

   (v) Find the eigenvalues of $D$. Is $D$ diagonalizable?
(vi) Find the eigenvalues of \( \partial \). Is \( \partial \) diagonalizable?

2. Prove that \( \mathbb{F}_2^\times \) is isomorphic to \( \mathbb{Z}/6\mathbb{Z} \), i.e. it is cyclic of order 6.

3. Let \( G \) be a group and let \( a, b \) be two elements in \( G \). Prove that \( ab \) and \( ba \) have the same order. (In other words, if \( ab \) has finite order, so \( ba \), and the two orders are equal.)

4. Let \( G = GL_2(\mathbb{R}) \), the group of invertible 2 \( \times \) 2 matrices with coefficients in \( \mathbb{R} \). Let

\[
H = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R}^\times \right\}
\]

be the subgroup of diagonal matrices in \( G \).

(i) Find the center \( Z(G) \) of \( G \).

(ii) Determine the centralizer \( Z_G(H) \) of \( H \) in \( G \).