1 Homework 11 Solutions

(1) Let $V$ be a vector space over a field not of characteristic 2. Define $\Lambda^2 V$ to be $V \otimes V / \text{Span}(v_i \otimes v_j + v_j \otimes v_i)$. Define $s(v \otimes w) = w \otimes v$ for any $v, w \in V$.

(a) Letting the $v_i$ be a basis for $V$ and denoting the equivalence class of $v_i \otimes v_j$ in the quotient by $v_i \wedge v_j$, show $\{v_i \wedge v_j\}_{i<j}$ forms a basis of $\Lambda^2 V$.

SOLUTION: The set $\{v_i \wedge v_j\}$ is a basis of $V \otimes V$, so the equivalence classes of these basis elements will be a basis of $\Lambda^2 V$. The quotienting relation $v_i \otimes v_j + v_j \otimes v_i = 0$ implies $v_i \wedge v_j = -v_j \wedge v_i$ in $\Lambda^2 V$. Thus for a given pair of basis elements $(v_i, v_j)$ we can order them with $i < j$ to obtain a basis element of $\Lambda^2 V$.

It follows that if $V$ has dimension $d$, there are $\frac{d(d-1)}{2}$ linearly independent $v_i \wedge v_j$ in $\Lambda^2 V$ which gives $\dim(\Lambda^2 V) = \frac{d(d-1)}{2}$.

(b) Show $s$ has eigenvalues 1 and -1 and is diagonalizable.

SOLUTION: Since $s^2$ is the identity transformation, the minimal polynomial of $s$ is $X^2 - 1$ which has distinct roots since the characteristic is not 2 (mod 2 this factors as $(X - 1)^2$ and thus does not have distinct roots); therefore $s$ is diagonalizable. In fact, it is not hard to check that the +1 eigenspace of $s$ is spanned by vectors of the form $v_i \otimes v_j + v_j \otimes v_i$ while the -1 eigenspace is spanned by vectors of the form $v_i \otimes v_j - v_j \otimes v_i$; it is easy to count the numbers of linearly independent vectors in these sets, thus showing that $s$ has a full set of eigenvectors and is therefore diagonalizable.

(c) Show that the projection $V \otimes V \to S^2 V$ gives isomorphisms $\ker(s-1) \cong S^2 V$ and $\ker(s+1) \cong \Lambda^2 V$.

SOLUTION: As mentioned in (b), a basis of $\ker(s-1)$ is elements of the form $s_{i,j} := v_i \otimes v_j + v_j \otimes v_i$; WLOG assume $i \leq j$. Suppose $k_1$ and $k_2$ are elements of $\ker(s-1)$; to show the projection gives an isomorphism we need to show that if $k_1$ and $k_2$ project to the same element of $S^2 V$ then $k_1 = k_2$. Letting $k_1 = \sum A_{i,j}s_{i,j}$ and $k_2 = \sum B_{i,j}s_{i,j}$. Since $S^2 V$ is obtained by setting $v_i \otimes v_j - v_j \otimes v_i = 0$ for all $i, j$, one has $p_{i,j} := \text{proj}(v_i \otimes v_j) = \text{proj}(v_j \otimes v_i)$; again we assume $i \leq j$. Thus $\text{proj}(s_{i,j}) = 2p_{i,j}$; note that the $p_{i,j}$ are linearly independent.

Now consider $\text{proj}(k_1 - k_2)$ which equals:

$$\text{proj}(k_1 - k_2) = \sum_{i \leq j} 2(A_{i,j} - B_{i,j})p_{i,j}$$

Because the $p_{i,j}$ are linearly independent, the only way this projection can be zero is if all coefficients $2(A_{i,j} - B_{i,j}) = 0$ which implies $A_{i,j} = B_{i,j} \forall i, j$, hence $k_1 = k_2$. An analogous argument shows the isomorphism $\ker(s+1) \cong \Lambda^2 V$. 

1
(2) Let $V$ be finite-dimensional over $\mathbb{C}$ and let $T \in \text{End}(V)$ such that $T^a = \text{Id}_V$ for some $a \geq 1$.

(a) Show $T$ is diagonalizable.

**SOLUTION:** We have encountered this problem numerous times. The minimal polynomial of $T$ divides $X^a - 1$, so has distinct roots, hence $T$ is diagonalizable.

(b) Suppose $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of $T$. Show:

\[
\text{Tr}(S^2(T)|_{S^2V}) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j
\]

\[
\text{Tr}(\Lambda^2(T)|_{\Lambda^2V}) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j
\]

**SOLUTION:** $T$ is diagonalizable hence has a full set of eigenvectors $v_1, \ldots, v_n$. By (2), a basis of $S^2(T)$ in $V \otimes V$ is $\{v_i \otimes v_j + v_j \otimes v_i\}_{i \leq j}$. Applying $T$ to one of these basis vectors one has:

\[
T(v_i \otimes v_j + v_j \otimes v_i) = Tv_i \otimes Tv_j + Tv_j \otimes Tv_i
= \lambda_i v_i \otimes \lambda_j v_j + \lambda_i v_j \otimes \lambda_i v_i
= \lambda_i \lambda_j (v_i \otimes v_j) + \lambda_j \lambda_i (v_j \otimes v_i)
= \lambda_i \lambda_j (v_i \otimes v_j + v_j \otimes v_i)
\]

Thus the eigenvalues of $T$ restricted to acting on $S^2V$ are as given. A similar argument holds for the eigenvalues of $T$ restricted to acting on $\Lambda^2V$. 

2
(3) Let $U$ be the irreducible complement of the trivial representation in the standard permutation action of $S_5$ acting on 5 points. Since a permutation action has character equal to the number of fixed points, one thusfar has the following characters of $S_5$ (here we have denoted the conjugacy classes by the lengths of the cycles in the disjoint cycle notation for the class, so $1^5$ is the identity class etc.):

<table>
<thead>
<tr>
<th>Class</th>
<th>$1^5$</th>
<th>$1^3 2^1$</th>
<th>$1^1 3^1$</th>
<th>$1^2 2^1$</th>
<th>$1^2 3^1$</th>
<th>$2^1 3^1$</th>
<th>$1^1 4^1$</th>
<th>$5^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>20</td>
<td>30</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>$\chi_{\text{triv}}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi_{\text{sgn}}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{\text{perm}}$</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\chi_U$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

(a) Calculate $\chi_{S^2 U}$.

SOLUTION: To use (2b) we need to know the eigenvalues of each class in the representation $U$; from the character we know the sum of these eigenvalues. Because $U$ is 4-dimensional, there are 4 eigenvalues that need to be determined for each class; by definition of a conjugacy class these 4 eigenvalues are the same for all elements in the class. Furthermore, in any group $G$ and any representation $\rho$, for an element $g$ of order $n$ one must have that the eigenvalues of $\rho(g)$ are all $n^{th}$ roots of unity (though not necessarily primitive) in order that $\rho(g)^n = \text{Id}$ can be satisfied. The sum of eigenvalues condition leads to the following possible eigenvalue sets ($\omega_j$ is a primitive $j^{th}$ root of unity):

<table>
<thead>
<tr>
<th>Class</th>
<th>Possible Eigenvalue Sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^5$</td>
<td>${1, 1, 1, 1}$</td>
</tr>
<tr>
<td>$1^3 2^1$</td>
<td>${1, 1, 1, -1}$</td>
</tr>
<tr>
<td>$1^1 3^1$</td>
<td>${1, 1, -1, -1}$</td>
</tr>
<tr>
<td>$2^1 3^1$</td>
<td>${-1, -1, \omega_6, \omega_6^5}$, ${1, -1, \omega_6, \omega_6^5}$, ${\omega_6, \omega_6^2, \omega_6^4, \omega_6^4}$, ${\omega_6, \omega_6^2, \omega_6^4, \omega_6^4}$</td>
</tr>
<tr>
<td>$1^1 4^1$</td>
<td>${1, 1, -1, -1}$, ${1, -1, \omega_4, \omega_4^3}$, ${\omega_4, \omega_4^3, \omega_4^3, \omega_4^3}$</td>
</tr>
<tr>
<td>$5^1$</td>
<td>${\omega_5, \omega_5^2, \omega_5^3, \omega_5^3}$</td>
</tr>
</tbody>
</table>

Note that the classes of elements of prime order there is only one possible eigenvalue set; this is a general feature of arbitrary finite groups: the character uniquely determines the corresponding set of eigenvalues for any element of prime order. On the other hand, for classes of elements of non-prime order there will usually be multiple possible eigenvalue sets; we now discuss how to pick the right such set.

For the two classes of elements not of prime order, we must determine which eigenvalue set is correct in order to be able to apply the result of (2b). How do we determine which set is correct for each of these classes? Well lets consider
the operation of raising an element to a power in general. If \( \rho(g) \) has eigenvalues \( \lambda_1, \ldots, \lambda_d \), then the eigenvalues of \( \rho(g^k) = \rho(g)^k \) are simply \( \lambda_1^k, \ldots, \lambda_d^k \).

Using this idea, let's try raising an element of class \( 2^13^1 \) to the second power; this gives an element of class \( 1^23^1 \) whose eigenvalue set we already know. Thus the squares of the eigenvalues of the class \( 2^13^1 \) form the set \( \{1, \omega_3, \omega_3^2, \omega_3^4 \} \).

Recalling that \( \omega_2^6 = \omega_3 \), we can thus rule out the sets \( \{\omega_6, \omega_6^2, \omega_6^3, \omega_6^4 \} \) and \( \{\omega_6^2, \omega_6^3, \omega_6^4, \omega_6^5 \} \) but not the other two. If we raise an element of class \( 2^13^1 \) to the third power then we are in class \( 1^32^1 \) so the cubes of its eigenvalues must form the set \( \{1, 1, -1, -1\} \) whereupon we see that the set \( \{1, -1, \omega_6^2, \omega_6^4\} \) is the only set satisfying both the square and cube conditions.

Similarly, elements of the class \( 1^14^1 \) square into class \( 1^12^2 \) whereupon the correct eigenvalue set for this class is \( \{1, -1, \omega_4, \omega_4^3\} \).

This procedure works in general; an arbitrary element \( g \in G \) can always be raised to some power \( k \) such that \( g^k \) has prime order; hence has uniquely determined eigenvalues in any representation. Some combination of these prime order powers and their eigenvalue sets will then be sufficient to determine the correct eigenvalue set of \( g \).

Having obtained the correct eigenvalue sets corresponding to the representation \( U \), we thus calculate the character of \( S^2U \) in two pieces (the reason for this is that the second piece goes ahead and calculates the character of \( \Lambda^2U \) which we want in (5)):

\[
\begin{array}{c|cc}
\text{Class} & \sum \lambda_i^2 & \sum_{i \neq j} \lambda_i \lambda_j \\
\hline
1^5 & 4 & 6 \\
1^32^1 & 4 & 0 \\
1^12^2 & 4 & -2 \\
1^23^1 & 1 & 0 \\
2^13^1 & 1 & 0 \\
1^14^1 & 0 & 0 \\
5^1 & -1 & 1 \\
\end{array}
\]

Table 1: Possible Eigenvalues of the Representation \( U \)

Adding these two columns gives \( \chi_{S^2U} \); using the inner product we then check that \( \chi_{S^2U} \) contains one copy each of \( \chi_{\text{triv}} \) and \( \chi_U \) and the remaining 5-dimensional character \( \chi_W \) is irreducible. This leads to the following characters for our table:

\[
\begin{array}{c|cccccccc}
\text{Class} & 1^5 & 1^32^1 & 1^12^2 & 1^23^1 & 2^13^1 & 1^14^1 & 5^1 \\
\hline
\chi_{S^2U} & 10 & 4 & 2 & 1 & 1 & 0 & 0 \\
\chi_W & 5 & 1 & 1 & -1 & 1 & -1 & 0 \\
\end{array}
\]

(c) Show \( W \) is irreducible.

SOLUTION: Already done in (3b).
(4a) Show $\chi_{\text{sgn}} \cdot \chi_U$ and $\chi_{\text{sgn}} \cdot \chi_W$ are irreducible characters not already calculated.

**SOLUTION:** Calculating these characters, one has:

<table>
<thead>
<tr>
<th>Class</th>
<th>$1^5$</th>
<th>$1^{3}2^1$</th>
<th>$1^{1}2^2$</th>
<th>$1^{2}3^1$</th>
<th>$2^{1}3^1$</th>
<th>$1^{1}4^1$</th>
<th>$5^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{\text{sgn}} \cdot \chi_U$</td>
<td>4</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_{\text{sgn}} \cdot \chi_W$</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

These are easily checked to be irreducible.

(b) Find the last character, $\chi_6$, via orthogonality relations.

**SOLUTION:** Since the sum of squares of the dimensions of the irreducible characters is 120, the final character is 6-dimensional (hence the name $\chi_6$). Our table of irreducible characters thusfar is:

<table>
<thead>
<tr>
<th>Class</th>
<th>$1^5$</th>
<th>$1^{3}2^1$</th>
<th>$1^{1}2^2$</th>
<th>$1^{2}3^1$</th>
<th>$2^{1}3^1$</th>
<th>$1^{1}4^1$</th>
<th>$5^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{\text{triv}}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{\text{sgn}}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_U$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_{\text{sgn}} \cdot \chi_U$</td>
<td>4</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_W$</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_{\text{sgn}} \cdot \chi_W$</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>6</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

Since we know the first column completely, we can now use column orthogonality with this column to find the final entry of the other six columns, leading to:

<table>
<thead>
<tr>
<th>Class</th>
<th>$1^5$</th>
<th>$1^{3}2^1$</th>
<th>$1^{1}2^2$</th>
<th>$1^{2}3^1$</th>
<th>$2^{1}3^1$</th>
<th>$1^{1}4^1$</th>
<th>$5^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_6$</td>
<td>6</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$\chi_6$ is easily checked to be irreducible.

(5) Show $\chi_{\Lambda_2U} = \chi_6$; this gives a realization of the final irreducible character as a representation of $S_5$.

**SOLUTION:** $\chi_{\Lambda_2U}$ was calculated as the second column of Table 1, so we see the two characters are equal.