1 Homework 6 Solutions

(1) Find a set of representatives of conjugacy classes in $GL_{10}(\mathbb{R})$ with characteristic polynomial $(x - 1)^4(x^2 + x + 1)^3$ and determine the minimal polynomial of each such class.

SOLUTION: Since every matrix is conjugate to its Jordan canonical form, we will consider the possible Jordan canonical forms of such matrices. The factors $(x - 1)$ and $(x^2 + x + 1)$ are irreducible over $\mathbb{R}$, so we first find matrices of size 1 and 2 (the degrees of the factors) with these factors as characteristic polynomial. The first matrix must be $M_1 = (1)$; we have more freedom with the second factor and $M_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ is one possibility.

With these matrices we create a basic Jordan form to represent the conjugacy classes:

\[
\begin{pmatrix}
M_1 & R_1 & 0 & 0 & 0 & 0 \\
0 & M_1 & R_2 & 0 & 0 & 0 \\
0 & 0 & M_1 & R_3 & 0 & 0 \\
0 & 0 & 0 & M_1 & 0 & 0 \\
0 & 0 & 0 & 0 & M_2 & S_1 \\
0 & 0 & 0 & 0 & 0 & M_2 & S_2
\end{pmatrix}
\]

Each $R_i$ and $S_i$ is either a zero matrix or identity matrix of the appropriate size. Since Jordan normal form depends only on the sizes of the blocks determined by the identity connectors and not their relative orders (changing the order of the blocks corresponds to a change of basis, hence is in the same conjugacy class),
we find that there are essentially only 5 distinct choices for the $R_i$:

<table>
<thead>
<tr>
<th>Jordan block structure of $M_1$ block</th>
<th>Minimal Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} M_1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; M_1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; M_1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; M_1 \end{pmatrix}$</td>
<td>$(x - 1)$</td>
</tr>
<tr>
<td>$\begin{pmatrix} M_1 &amp; Id_1 &amp; 0 &amp; 0 \ 0 &amp; M_1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; M_1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; M_1 \end{pmatrix}$</td>
<td>$(x - 1)^2$</td>
</tr>
<tr>
<td>$\begin{pmatrix} M_1 &amp; Id_1 &amp; 0 &amp; 0 \ 0 &amp; M_1 &amp; Id_1 &amp; 0 \ 0 &amp; 0 &amp; M_1 &amp; Id_1 \ 0 &amp; 0 &amp; 0 &amp; M_1 \end{pmatrix}$</td>
<td>$(x - 1)^3$</td>
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<tr>
<td>$\begin{pmatrix} M_1 &amp; Id_1 &amp; 0 &amp; 0 \ 0 &amp; M_1 &amp; Id_1 &amp; 0 \ 0 &amp; 0 &amp; M_1 &amp; Id_1 \ 0 &amp; 0 &amp; 0 &amp; M_1 \end{pmatrix}$</td>
<td>$(x - 1)^4$</td>
</tr>
<tr>
<td>$\begin{pmatrix} M_1 &amp; Id_1 &amp; 0 &amp; 0 \ 0 &amp; M_1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; M_1 &amp; Id_1 \ 0 &amp; 0 &amp; 0 &amp; M_1 \end{pmatrix}$</td>
<td>$(x - 1)^2$</td>
</tr>
</tbody>
</table>

Similarly there are essentially 3 distinct choices for the $S_i$:

<table>
<thead>
<tr>
<th>Jordan block structure of $M_2$ block</th>
<th>Minimal Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} M_2 &amp; 0 &amp; 0 \ 0 &amp; M_2 &amp; 0 \ 0 &amp; 0 &amp; M_2 \end{pmatrix}$</td>
<td>$(x^2 + x + 1)$</td>
</tr>
<tr>
<td>$\begin{pmatrix} M_2 &amp; Id_2 &amp; 0 \ 0 &amp; M_2 &amp; 0 \ 0 &amp; 0 &amp; M_2 \end{pmatrix}$</td>
<td>$(x^2 + x + 1)^2$</td>
</tr>
<tr>
<td>$\begin{pmatrix} M_2 &amp; Id_2 &amp; 0 \ 0 &amp; M_2 &amp; Id_2 \ 0 &amp; 0 &amp; M_2 \end{pmatrix}$</td>
<td>$(x^2 + x + 1)^3$</td>
</tr>
</tbody>
</table>
(2) Let \( T \in \text{End}_F(V) \) have minimal polynomial \( f \in F[x] \) and let \( R(T) \) be the subring of \( \text{End}_F(V) \) generated by \( T \), i.e. elements of \( R \) are single-variable polynomials with the variable replaced by \( T \).

(a) Suppose \( f \) is divisible by \( g^2 \) for \( g \) some irreducible polynomial. Show \( R \) contains nontrivial nilpotent elements.

SOLUTION: Let \( f(x) = g(x)^2h(x) \) for some \( h \in F[x] \). The element \( g(T)h(T) \) is nonzero since \( gh \) is not divisible by the minimal polynomial of \( T \). But 
\[
(g(T)h(T))^2 = g(T)h(T)g(T)h(T) = g(T)^2h(T)^2 = f(T)h(T) = 0,
\]
so \( R \) contains a nontrivial nilpotent element.

(b) Suppose \( f \) is squarefree. Show 0 is the only nilpotent element of \( R \).

SOLUTION: \( F[x] \) is a UFD since \( F \) is a field. Suppose \( \exists p \in F[x] \) such that \( p(T)^m = 0 \) for some \( m \geq 2 \) but \( p(T) \neq 0 \). Thus \( f(x) \) divides \( p(x)^m \), hence every irreducible factor of \( f(x) \) must be an irreducible factor of \( p(x)^m \). As \( f \) is squarefree, \( \text{deg}(p) \geq \text{deg}(f) \) whereupon \( f(x) \) divides \( p(x) \), a contradiction.
Let $G$ be a group with 6 elements and $u$ an element of order 3, and $V = \mathbb{C}[G]$ (as a vector space, $V$ is simply isomorphic to $\mathbb{C}^6$). Let $T \in \text{End}_\mathbb{C}(V)$ be given by left multiplication by $u$.

(a) Show that as a $\mathbb{C}[x]$-module with the $x$-action on $V$ given by $T$ (essentially $x$ becomes the element $u$), $V \cong \mathbb{C}[x]/(x^3 - 1) \oplus \mathbb{C}[x]/(x^3 - 1)$.

SOLUTION: It will not matter which group of order 6 we choose, so we'll work with $\mathbb{Z}/6\mathbb{Z}$ for convenience. Let $u = 2$, then (left) multiplication by $u$ acts separately on two 3 dimensional subspaces:

$$\begin{align*}
\cdots &\rightarrow 0 \rightarrow 2 \rightarrow 4 \rightarrow 0 \rightarrow \cdots \\
\cdots &\rightarrow 1 \rightarrow 3 \rightarrow 5 \rightarrow 1 \rightarrow \cdots
\end{align*}$$

$T$ therefore acts as an element of order 3 on the two subspaces $\{0, 2, 4\}$ and $\{1, 3, 5\}$ whereupon the result follows.

(b) Let $C$ be the subring of $\text{End}_\mathbb{C}(V)$ of $\mathbb{C}$-endomorphisms which commute with $T$. Is $C$ commutative?

SOLUTION: No, $C$ is not commutative. Relative to the ordered basis $\{0, 2, 4, 1, 3, 5\}$, $T$ has matrix:

$$T = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

Note that $T$ has a block form $T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ where $t = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. $C$ consists of all matrices $R$ such that $RT = TR$. It is not hard to check that for complex numbers $c_1 \ldots c_4$, the $6 \times 6$ matrix \(\begin{pmatrix} c_1 t & c_2 t \\ c_3 t & c_4 t \end{pmatrix}\) commutes with $T$. But it is also easily checked that two such matrices commute iff the corresponding $2 \times 2$ matrices obtained by deleting the $t$-factors commute, hence $C$ is not commutative.

(c) Find $\dim_\mathbb{C}(C)$.

SOLUTION: Using the commuting condition, it is not hard to check that $T$ only commutes with matrices of the block form $\begin{pmatrix} a_1 + a_2 t + a_3 t^2 & b_1 + b_2 t + b_3 t^2 \\ c_1 + c_2 t + c_3 t^2 & d_1 + d_2 t + d_3 t^2 \end{pmatrix}$ with $t$ as above. Since there are 12 free parameters in this form, $\dim_\mathbb{C}(C) = 12$. 

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(4) Let $A \in \text{Mat}_{n \times n}(F)$ with $F$ a field. Show $A^t$ is conjugate to $A$.

**SOLUTION:** This follows easily from the following lemma:

**Lemma:** A Jordan block is conjugate to its transpose.

**Proof:** Let $J_m(c)$ be an $m \times m$ Jordan block with eigenvalue $c$ and let $U_m$ be the $m \times m$ matrix with 1’s down the antidiagonal and 0’s elsewhere:

$$U_m = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}$$

It is easy to check that $U_m J_m(c) = J_m(c) U_m$ and $U_m$ is clearly equal to its own inverse, hence $U_m J_m(c) U_m^{-1} = J_m(c)^t$ which proves the lemma.

Now suppose that $A$ is arbitrary and $J_A$ is the Jordan normal form of $A$. Then $\exists C$ such that $CAC^{-1} = J_A$. Applying transposes to this equality, one has $(C^{-t}) A^t C^t = J_A^t$ where $C^{-t}$ denotes the inverse of $C^t$. Thus $A^t$ is conjugate to $J_A^t$; by the above lemma applied to the blocks one has $J_A = U_A J_A^t U_A^{-1}$ with $U_A = \begin{pmatrix}
U_{m_1} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & U_{m_k}
\end{pmatrix}$ with the $m_i$ the sizes of the blocks of $J_A$. Putting this all together, one explicitly has the conjugacy between $A$ and $A^t$:

$$(C^t U_A C)(C^t U_A C)^{-1} = A^t$$