1. Let $V, W$ be vector spaces over a field $F$. Prove that there exists a unique isomorphism $s: V \otimes_F W \rightarrow W \otimes_F V$ such that $s(v \otimes w) = w \otimes v$ for all $v \in V$ and all $w \in W$.

2. Let $F$ be a field and let $U, V, W$ be $F$-vector spaces.
   
   (a) Let $\beta: U \times V \times W \rightarrow U \otimes_F (V \otimes_F W)$ be the map given by $\beta(u, v, w) = u \otimes (v \otimes w)$.
   Show that $\beta$ is $F$-trilinear, i.e. it is $F$-linear in $U$, $V$, and $W$ separately.
   
   (b) Let $X$ be an $F$-vector space and let $T: U \times V \times W \rightarrow X$ be an $F$-trilinear map. Prove that there exists a unique $F$-linear map $f: U \otimes_F (V \otimes_F W) \rightarrow X$ such that $T = f \circ \beta$, where $\beta$ is the map defined in (a).
   
   (c) (extra credit) Prove that there exists a unique $F$-linear isomorphism $\alpha: U \otimes_F (V \otimes_F W) \rightarrow (U \otimes_F V) \otimes_F W$ such that $\alpha(u \otimes (v \otimes w)) = (u \otimes v) \otimes w$ for all $u \in U$, all $v \in V$ and all $w \in W$.
   [Hint: One way is to use the universal property of $\beta$ proved in (b) and a similar property for $(U \otimes_F V) \otimes_F W$.]

3. Let $(\rho, V)$ be a two-dimensional irreducible $\mathbb{C}$-linear representation of the symmetric group $S_3$.
   
   (a) Compute explicitly the character of the tensor product representation $(\rho \otimes \rho, V \otimes V)$ of $S_3$.
   
   (b) Decompose explicitly the character of $\rho \otimes \rho$ into a sum of irreducible characters (with multiplicity).

4. Let $V$ be a vector space over a field $F$. Let $S^2(V)$ be the quotient of $V \otimes_F V$ by the $F$-linear span $U$ of all elements in $V \otimes_F V$ of the form $v_1 \otimes v_2 - v_2 \otimes v_1$, $v_1, v_2 \in V$.
   (This vector space $S^2(V) = V \otimes_F V/U$ is called the second symmetric product of $V$.)
(a) Suppose that $\dim_{F}(V) = 2$. Prove that $\dim_{F}(S^{2}(V)) = 3$.

(b) (extra credit) Find a general formula for the dimension of $S^{2}(V)$ in terms of the dimension of $V$.

5. (extra credit)

(a) Suppose that $(\rho, V)$ is a $\mathbb{C}$-linear representation of a finite group $G$. Show that the $\mathbb{C}$-linear subspace $U$ of $V \otimes V$ generated by all elements of the form $v_{1} \otimes v_{2} - v_{2} \otimes v_{1}$ with $v_{1}, v_{2} \in V$ is stable under the action of $G$, so that we get a natural linear representation of $G$ on the quotient $S^{2}(V) = V \otimes V / U$, called the second symmetric product of $(\rho, V)$.

(b) Compute the character of the second symmetric product of the 2-dimensional irreducible representation of the quaternion group $Q$. 