The goal of this set of problems (including the extra credit ones) is to compute the character table of the symmetric group $S_5$ “by hand”, and construct some irreducible representations of $S_5$ to the point that we can figure out the whole character table. Many things we have learned before will come handy, including:

- canonical forms, especially diagonalizable linear operators,
- the permutation representation attached to the action of a finite group on a finite set,
- the second symmetric product of a representation

We will also discuss the second exterior product of a vector space; it is useful for constructing a 6-dimensional irreducible representation of $S_5$.

The group $S_5$ has 120 elements, and 7 conjugacy classes. A set of representatives of these seven conjugacy classes are: $e$, $(12)$, $(12)(34)$, $(123)$, $(123)(45)$, $(1234)$, $(12345)$; the cardinality of the respective conjugacy classes are 1, 10, 15, 20, 20, 30 and 24.

We have a natural action of $S_5$ on the set $S = \{1, 2, 3, 4, 5\}$ of 5 letters. This action is **doubly transitive**. Let $W$ be the $\mathbb{C}$-vector space of all $\mathbb{C}$-valued functions on $S$. The natural action of $S_5$ on $W$ via linear permutation representations decomposes into the direct sum of the trivial representation $1$ and a complement $U$ of $1$ in $W$ consisting of all functions $f$ on $S$ such that $\sum_{x \in S} f(x) = 0$. We know from HW 9, problem 1 that $U$ is an irreducible representation of $S_5$; denote this representation by $\rho_3$. Let $\chi_2 : S_5 \to \mathbb{C}^\times$ be the sign character of $S_5$. Recall that for every linear representation $\rho$ of $S_5$, $\chi_2 \cdot \rho$ is another linear representation of $S_5$, irreducible if $\rho$ is.

1. (This is a continuation of problem 4 of HW 10.) Let $V$ be a vector space over a field $F$. The **second exterior product** $\Lambda^2_F(V)$ of $V$ is by definition the quotient of $V \otimes_F V$ by the subspace of $V \otimes_F V$ spanned by all elements of the form

$$v_1 \otimes v_2 + v_2 \otimes v_1,$$

$v_1, v_2 \in V$.

The image of $v_1 \otimes v_2$ in the quotient $F$-vector space $\Lambda^2(V)$ is denoted $v_1 \land v_2$, for all $v_1, v_2 \in V$.

(a) Suppose that $v_1, \ldots, v_d$ is an $F$-basis of $V$. Show that

$$\{v_i \land v_j \mid 1 \leq i < j \leq d\}$$

form an $F$-basis of $\Lambda^2_F(V)$. Conclude that $\dim_F(\Lambda^2_F(V)) = d(d-1)/2$.
Recall from problem 1 of HW 10 that there exists a unique linear operator \( s \in \text{End}_F(V \otimes V) \) such that
\[
s(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \text{for all } v_1, v_2 \in V.
\]

(b) Assume that \( 2 \in F^\times \), i.e., the characteristic of the field \( F \) is not 2. Assume moreover that \( \dim_F(V) \geq 2 \), for otherwise \( \Lambda^2(V) = (0) \). Prove that \( s \) is diagonalizable and has 1 and \(-1\) as its eigenvalues.

(c) Let \( \text{Sym}(V \otimes V) = \text{Ker}(s - 1_{V \otimes V}) \) and let \( \text{Alt}(V \otimes V) = \text{Ker}(s + 1_{V \otimes V}) \). Prove that the projection from \( V \otimes V \) to \( S^2(V) \) induces an \( F \)-linear isomorphism \( \text{Sym}(V \otimes V) \sim \rightarrow S^2(V) \), and that the projection from \( V \otimes V \) to \( \Lambda^2(V) \) induces an \( F \)-linear isomorphism \( \text{Alt}(V \otimes V) \sim \rightarrow \Lambda^2(V) \).

Note that if a group \( G \) operates \( F \)-linearly on \( V \), then we have an induced action of \( G \) on \( S^2(V) \) and on \( \Lambda^2(V) \), called the second symmetric product and the second exterior product of the representation of \( G \) on \( V \).

2. Let \( T \in \text{End}_C(V) \) be a linear operator on a finite dimensional \( C \)-vector space \( V \). Assume that \( T^a = \text{Id}_V \) for some positive integer \( a \geq 1 \).
   (a) Prove that \( T \) is diagonalizable.
   (b) Suppose that \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( T \) listed with multiplicity. Show that
   \[
   \text{Tr}(S^2(T)|_{S^2(V)}) = \sum_{1 \leq i \leq j \leq n} \lambda_i \cdot \lambda_j
   \]
   and
   \[
   \text{Tr}(\Lambda^2(T)|_{\Lambda^2(V)}) = \sum_{1 \leq i < j \leq n} \lambda_i \cdot \lambda_j.
   \]
   This formula will be convenient for computing the character of the second symmetric product (resp. the second exterior product) of a given representation.

3. Let \( U \) be the irreducible four-dimensional representation of \( S_5 \), defined to be the complement of the trivial representation in the standard permutation representation of \( S_5 \).
   (a) Compute explicitly the character the action of \( S_5 \) of \( S^2(U) \).
   (b) Prove that the trivial representation \( 1 \) and \( U \) both appears in \( S^2(U) \) with multiplicity 1, so we have a \( S_5 \)-linear isomorphism
   \[
   S^2(U) \cong C \oplus U \oplus W,
   \]
   where \( W \) is a 5-dimensional-linear representation of \( S_5 \). Let \( \chi_5 \) be the character of the \( S_5 \)-action on \( W \).
   (c) Compute the character of \( \chi_5 \) and show that \( S_5 \) operates irreducibly on \( W \).
4. (extra credit) So far we have four irreducible representations: \( \chi_1 = 1 \), \( \chi_2 = \text{sgn} \), the four dimensional character \( \chi_3 \) and the 5-dimensional character \( \chi_5 \).

(a) Show that \( \chi_4 := \text{sgn} \cdot \chi_3 \) and \( \chi_6 := \text{sgn} \cdot \chi_5 \) are irreducible characters of \( S_4 \), different from the above four characters.

(b) Compute the seventh irreducible character \( \chi_7 \) of \( S_5 \) and complete the character table of \( S_5 \).

5. (extra credit) Compute the character of the representation of \( S_5 \) on \( \Lambda^2(\mathbb{U}) \) and show that it is an irreducible character. (So we have found the irreducible representation of \( S_5 \) with character \( \chi_7 \).)