1. Prove that the quotient ring \( S := \mathbb{Q}[x, y]/(x^2 - y^2 - 1) \) of the polynomial ring \( \mathbb{Q}[x, y] \) is an integral domain.
   (Please write a complete proof.)

2. (a) Factor the polynomial \( x^4 - y^4 \) into a product of irreducible elements in \( \mathbb{Q}[x, y] \) and in \( \mathbb{C}[x, y] \).
   (You need show that factors you claim to be irreducible are indeed so.)

   (b) (extra credit) Factor the polynomial \( x^3 + y^3 + z^3 - 3xyz \) into a product of irreducible elements in \( \mathbb{C}[x, y, z] \).

3. Is the group ring \( \mathbb{Q}[\mathbb{Z}/5\mathbb{Z}] \) of the cyclic group \( \mathbb{Z}/5\mathbb{Z} \) an integral domain?
   (Please write a complete proof.)

4. Let \( R := \mathbb{Q}[u, u^{-1}] \) be the subring of of the fraction field \( \mathbb{Q}(u) \) of the polynomial ring \( \mathbb{Q}[u] \), consisting of all elements in \( \mathbb{Q}(u) \) of the form
   \[
   \frac{f(u)}{u^n}, \quad f(u) \in \mathbb{Q}[u], \ n \in \mathbb{N}.
   \]
   Prove that \( R \) is a principal ideal domain.
   (Prove that if \( I \) is an ideal of \( R \) and \( g(u) \) is a generator of the ideal \( I \cap \mathbb{Q}[u] \) of \( \mathbb{Q}[u] \), then \( g(u) \) generates the ideal \( I \) in \( R \).)

5. (Extra Credit)

   (a) Let \( S \) be the ring in problem 1 above. Let \( \bar{x}, \bar{y} \) be the image of \( x \) and \( y \) in the quotient ring \( S \) of \( \mathbb{Q}[x, y] \). Show that the ideal \( (\bar{x} - 1, \bar{y}) \) in \( S \) is a principal ideal, and find a generator of this ideal.

   (b) Prove that the ring \( S \) in problem 1 is isomorphic to the ring \( R \) in problem 4. Conclude that \( S \) is a principal ideal domain.