1. Give an example of a UFD which is not a PID.

2. Give an example of a commutative ring which is not an integral domain and has only one maximal ideal.

3. Give an example of a maximal ideal in a commutative ring which is not a prime ideal.

4. Let $R$ be a UFD and let $x$ be an irreducible element of $R$.
   
   (a) Is $(x)$ a maximal ideal?
   
   (b) Is $(x)$ a prime ideal?
   
   (c) Is every maximal ideal of $R$ principal?

5. Let $R = \mathbb{R}[\mathbb{Z}/5\mathbb{Z}]$ and let $M = R/R \cdot (-[\bar{0}] + [\bar{1}])$, where $\bar{0}$ is the unity element of the cyclic group $\mathbb{Z}/5\mathbb{Z}$, $\bar{1}$ is the image of $1 \in \mathbb{Z}$ in $\mathbb{Z}/5\mathbb{Z}$, and $R \cdot (-[\bar{0}] + [\bar{1}])$ is the left ideal of $R$ generated by $-\bar{0} + \bar{1}$.

   (a) Find all $R$-module endomorphisms of $M$.
   
   (b) Is the ring $\text{End}_R(M)$ commutative?
   
   (c) Is the ring $\text{End}_R(M)$ an integral domain?

6. Let $G$ be a finite group and let $\rho : G \to \text{GL}_n(\mathbb{C})$ be an $n$-dimensional complex linear representation of $G$. Prove that the matrix $\rho(g)$ is diagonalizable for every $g \in G$.

   (Hint: The minimal polynomial of $\rho(g)$ divides $x^{|\text{Card}(G)|} - 1$.)

7. Is it true that every $5 \times 5$ matrix $A \in M_5(\mathbb{R})$ is conjugate to an upper-triangular matrix in $M_5(\mathbb{R})$ by an element $C \in \text{GL}_5(\mathbb{R})$?

8. Let $N_1$ and $N_2$ be two $n \times n$ nilpotent matrices in $M_n(F)$, where $F$ is a field. Assume that $N_1$ and $N_2$ have the same minimal polynomial.

   (a) Suppose that $n = 3$. Are $N_1$ and $N_2$ conjugate?
   
   (b) Suppose that $n = 4$. Are $N_1$ and $N_2$ conjugate?
   
   (c) Suppose that $n = 6$ and $\dim_F(\ker(N_1)) = \dim_F(\ker(N_2))$. Are $N_1$ and $N_2$ conjugate?
   
   (d) Suppose that $n = 7$ and $\dim_F(\ker(N_1)) = \dim_F(\ker(N_2))$. Are $N_1$ and $N_2$ conjugate?
9. Let $S$ be the square in $\mathbb{R}^2$ with vertices $\{(\pm 1, \pm 1)\}$. The group of all symmetries of $S$ is the dihedral group $D_8$ with 8 elements; its action on $S$ extends uniquely to a linear action of $D_8$ on $\mathbb{R}^2$. This linear action gives $\mathbb{R}^2$ the structure of a left module over the group ring $\mathbb{R}[D_8]$. What is the ring $\text{End}_{\mathbb{R}[D_8]}(\mathbb{R}^2)$?

(Note that $\text{End}_{\mathbb{R}[D_8]}(\mathbb{R}^2)$ consists of all $\mathbb{R}$-linear operators on $\mathbb{R}^2$ which commute with the $D_8$-action.)

10. Classify all commutative groups with 1,000 elements. How many isomorphism classes of such groups are there?