1. Let $\mathcal{F}: \text{Ab} \to \text{Sets}$ be the forgetful functor from the category of abelian groups to the category of sets, which sends an abelian group $A$ to the set underlying $A$.

(a) Prove that the above forgetful functor has a left adjoint, and describe it explicitly.

[The answer was indicated in class.]

(b) Either prove that $\mathcal{F}$ has a right adjoint, or show that no such right adjoint exists.

2. Do problem 29 of Gallier-Shatz.

[Notes:
1. The two functors $|M_{pq}|$ and $|GL_n|$ in subproblem 1 of this problem are covariant functors from the category of commutative rings with 1 to the category of sets.
2. For subproblem 4, you need to describe the functor $I$ explicitly, and either prove the existence of a left adjoint of $I$ or prove that $I$ does not have a left adjoint.]

3. Let $\mathcal{G}_m: \text{Rings} \to \text{Groups}$ be the functor from the category of (non-trivial) rings with 1 to the category of groups, which sends each ring $R$ to the group $R^\times$ of all units in $R$. Either show that $\mathcal{G}_m$ has a right adjoint, or show that no such right adjoint exists.

4. Let $\text{Rings}$ be the category of rings with 1; recall that $0 \neq 1$ in a ring with 1 by convention. and let $\text{Rings}^+$ be the category of possibly trivial rings with 1, in the sense that $0 = 1$ is allowed. Note that $\mathbb{Z}$ is an initial object in $\text{Rings}^+$, and any trivial ring is a final object in $\text{Rings}^+$.

(a) Show that non-empty products (not necessarily finite) exist in the category $\text{Rings}^+$.

(b) Let $\mathcal{F}: \text{Rings}^+ \to \text{Sets}$ be the forgetful functor from the category $\text{Rings}^+$ to the category of sets, which sends every (possibly trivial) ring $R$ to the set underlying $R$. Prove that $\mathcal{F}$ has a left adjoint, and describe it explicitly.

(c) Do coproducts exist in the category $\text{Rings}^+$? Either prove this statement or give a counterexample.