1. Let $K/F$ be an extension of finite fields. Show that the norm map $\text{Nm}_{K/F}$ and the trace map $\text{Tr}_{K/F}$, as maps from $K$ to $F$, are both surjective. (The relative norm and trace of an element $x \in K$ are the determinant and trace of the $F$-linear transformation “multiplication by $x$” on the $F$-vector space $K$.)

2. Let $K = \mathbb{F}_q(t)$, the rational function field over $\mathbb{F}_q$ in one variable $t$. The group $\text{PGL}_2(\mathbb{F}_q)$ operates on $K$ via linear fractional transformations: an element represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $t$ to $\frac{at+b}{ct+d}$.

Show that the subfield of $K$ fixed by $\text{PGL}_2(\mathbb{F}_q)$ is $\mathbb{F}_q(u)$, where

$$u = \frac{(t^q - t)^{q+1}}{(t - 1)^{q+1}}.$$

3. Prove the following version of algebraic independence of characters. Let $K$ and $L$ be infinite fields, and let $\sigma_1, \ldots, \sigma_n$ be distinct ring homomorphisms from $K$ to $L$. If $\text{char}(K) = p > 0$, assume that $\sigma_i \neq \sigma_j^p$ for all $i \neq j$ and all $r \in \mathbb{N}$. If $f(T_1, \ldots, T_n) \in L[T_1, \ldots, T_n]$ is a polynomial with coefficients in $L$ such that $f(\sigma_1(\xi), \ldots, \sigma_n(\xi)) = 0$ for all $\xi \in K$, then $f(T_1, \ldots, T_n) = 0$. (The condition on the embeddings $\sigma_1, \ldots, \sigma_n$ is clearly necessary: For instance if $\sigma_1 = \sigma_2^p$, then $f(T_1, T_2) = T_1 - T_2^p$ is an algebraic relation between $\sigma_1$ and $\sigma_2$.)

[Hint: Suppose that $\sigma_1, \ldots, \sigma_n$ are algebraically dependent over $L$. Let $g(T_1, \ldots, T_n)$ be a non-trivial algebraic relation over $L$ satisfied by all $n$-tuples of the form $(\sigma_1(\xi), \ldots, \sigma_n(\xi))$, with $\xi \in K$ of lowest possible degree. Show first that $g(T_1, \ldots, T_n)$ is additive in the sense that

$$g(U_1 + V_1, \ldots, U_n + V_n) = g(U_1, U_2, \ldots, U_n) + g(V_1, V_2, \ldots, V_n)$$

in the polynomial ring $L[T_1, \ldots, T_n]$. Consequently $g(T_1, T_2, \ldots, T_n) = f_1(T_1) + f_2(T_2) + \cdots + f_n(T_n)$ for suitable additive polynomials $f_1(X), \ldots, f_n(X) \in L[X]$.)


5. Do Gallier/Shatz problem 118. Note that in part 1 of this problem the number 2 is quite special. Suppose that a field extension $K/F$ can be obtained from $F$ by a succession of cubic extensions. Let $L$ be the smallest normal extension of $F$ which contains $K$. What can you say about $[L : F]$ and $\text{Gal}(L/F)$?