1. (a) Let $K/F$ be a finite Galois extension. Let $V$ be a finite dimensional $K$-vector space. Suppose we are given a $\mathbb{Z}$-linear action of $\Gamma := \text{Gal}(K/F)$ on $V$ such that

$$\sigma(x \cdot v) = \sigma(x) \cdot \sigma(v) \quad \forall x \in K, \forall v \in V.$$ 

Let $W := V^G = \{v \in V \mid \sigma(v) = v \ \forall \sigma \in \Gamma\}$. Show that the natural map from $K \otimes_F W$ to $V$ is a $K$-linear isomorphism.

(b) Show that the assumption that $\dim_K(V) < \infty$ in (a) can be eliminated.

(c) Let $K/F$ be a finite Galois extension as in (a). Let $R$ be a $K$-algebra (not necessarily commutative) with an action by $\Gamma$ such that every element $\Gamma$ operates as a ring automorphism of $R$ and

$$\sigma(x \cdot r) = \sigma(x) \cdot \sigma(r) \quad \forall x \in K, \forall r \in R.$$ 

Show that there exists an $F$-algebra $S$ and a $\Gamma$-equivariant isomorphism

$$\phi : K \otimes_F S \xrightarrow{\sim} R,$$

where $\Gamma$ acts on the tensor product $K \otimes_F S$ through its action on $K$. Prove also that the pair $(S, \phi)$ is determined up to unique isomorphism.

2. Let $G$ be a finite group and let $p$ be a prime number which divides $\text{Card}(G)$. Let $k$ be a field of characteristic $p$. Prove that the group ring $k[G]$ is not semisimple.

3. Give an example of a central division algebra $D$ over a field $K$ such that $\dim_K(D) = \infty$.

4. Let $R$ be a ring. The radical $N$ of $R$ is by definition the intersection of all maximal left ideals of $R$.

(a) Show that $N \cdot S$ for every simple left $R$-module $S$.

(b) Suppose that $I$ is a left ideal of $R$ such that $I \cdot S$ for every simple $R$-module $S$. Prove that $S \subseteq N$.

(c) Let $x$ be an element of the radical $N$. Show that $R \cdot (1+x) = R$. Let $y$ be an element of $R$ such that $-x = y \cdot (1+x)$. Prove that $y \in R$ and $1+y$ is the inverse of $1+x$.

(d) Prove that $N$ is contained in every maximal right ideal of $R$. Conclude that $N$ is equal to the intersection of all maximal right ideals of $R$.

(e) Prove the following non-commutative analogue of Nakayama’s lemma. Let $M$ be a finitely generated $R$ module. If $R \cdot M \subseteq N \cdot M$, then $M = (0)$.

5. Let $D$ be a division ring. Let $R$ be the set of all matrices whose rows and columns are indexed both indexed by $\mathbb{N}$ and every column has only finitely many non-zero entries. Clearly $R$ is stable under matrix addition and multiplication, therefore $R$ has a natural ring structure.

(a) Determine the radical of this ring $R$.

(b) Find a simple left $R$-module $M$.

(c) Show that every $R$-module is a direct sum of copies of $M$. 