1. Give an example of two finite group \( G_1, G_2 \) and irreducible linear representations \( (\rho_i, V_i) \) over \( \mathbb{Q} \) of \( G_i \) for \( i = 1, 2 \) such that the external tensor product representation \( \rho_1 \boxtimes \rho_2 \) of \( G_1 \times G_2 \) on \( V_1 \otimes_{\mathbb{Q}} V_2 \) is not irreducible over \( \mathbb{Q} \).

2. Recall that for every finite set \( S \) with a left action by a finite group \( G \), we have a natural representation \( \rho_{(G,S)} \) on the set \( C(S; \mathbb{C}) \) of all \( \mathbb{C} \)-valued functions on \( S \). The action is

\[
(\rho_{(G,S)}(g)(f))(s) = f(g^{-1} \cdot s) \quad \forall g \in G, \forall f \in C(S; \mathbb{C}), \forall f \in C(S; \mathbb{C}).
\]

(a) Show that \( \langle \chi_{\rho_{(G,S)}} \rangle|_1 = \text{Card}(G \setminus S) \). Here \( 1 \) stands for the trivial character of \( G \).

(b) Let \( U := \{ f \in C(S; \mathbb{C}) \mid \sum_{s \in S} f(s) = 0 \} \), a \( G \)-subrepresentation of \( C(S; \mathbb{C}) \). Assume that \( \text{Card}(S) \geq 2 \), so \( U \) is non-trivial, and \( G \) operates transitively on \( S \). Prove that \( U \) is an irreducible representation if and only if \( G \) operates doubly transitively on \( S \). (Recall that \( G \) operates doubly transitively on \( S \) if for any two pairs \( (s, t) \) and \( (s', t') \) with \( s, t, s', t' \in S \), \( s \neq t \) and \( s' \neq t' \), there exists an element \( g \in G \) such that \( (g \cdot s, g \cdot t) = (s', t') \).

[Hint: Use (a).]

3. Let \( G \) be a finite group. Let \( n \geq 2 \) be a positive integer, let \( m \) be an integer prime to \( n \), and let \( x \in G \) be an element such that \( x^n = e \). Let \( \chi \) be the character of a finite dimensional \( \mathbb{C} \)-representation of \( G \)

(a) Show that \( \chi(x) \in \mathbb{Q}(\mu_n) \), the cyclotomic field generated by all \( n \)-th roots of 1.

(b) Let \( \sigma_m \in \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \) be the automorphism of \( \mathbb{Q}(\mu_n) \) such that \( \sigma_m(\zeta) = \zeta^m \) for all \( z \in \mu_n \). Show that \( \chi(x^m) = \sigma_m(\chi(x)) \). (This is another constraint on the character table. In the case \( m = -1 \) this is a constraint we discussed in class, that \( \chi(x^{-1}) \) is the complex conjugate of \( \chi \).

4. The goal of this problems is to compute the character table of the symmetric group \( S_5 \) “by hand”, and construct some irreducible representations of \( S_5 \) to the point that we can figure out the whole character table. The group \( S_5 \) has 120 elements, and 7 conjugacy classes. A set of representatives of these seven conjugacy classes are: \( e \), \( (12) \), \( (12)(34) \), \( (123) \), \( (123)(45) \), \( (1234) \), \( (12345) \); the cardinality of the respective conjugacy classes are 1, 10, 15, 20, 20, 30 and 24.

We have a natural action of \( S_5 \) on the set \( S = \{1, 2, 3, 4, 5\} \) of 5 letters. Let \( W = C(S; \mathbb{C}) \) be the \( \mathbb{C} \)-vector space of all \( \mathbb{C} \)-valued functions on \( S \), with natural action of \( S_5 \) on \( W \) via linear permutation representations; \( W \) decomposes into the direct sum of the trivial representation \( 1 \) and a complement \( U \) as in problem 2 above. We know from problem 2 that \( U \) is an irreducible representation of \( S_5 \); denote this representation by \( \rho_3 \). Let \( \chi_2 : S_5 \to \mathbb{C}^\times \) be the sign character of \( S_5 \). Note that for every linear representation \( \rho \) of \( S_5 \), \( \chi_2 \cdot \rho \) is another linear representation of \( S_5 \), irreducible if \( \rho \) is. (Prove it!)

We will use the following general construction. For any linear representation \( V \) of a group \( G \), tensor constructions in \( V \) are again representations of \( G \). They include the symmetric powers \( S^n(V) \) and exterior powers \( \wedge^n(V) \).
(a) Compute explicitly the character the action of $S_5$ of $S^2(U)$.

(b) Prove that the trivial representation $\mathbf{1}$ and $U$ both appears in $S^2(U)$ with multiplicity 1, so we have a $S_5$-linear isomorphism

$$S^2(U) \cong \mathbb{C} \oplus U \oplus W,$$

where $W$ is a 5-dimensional-linear representation of $S_5$. Let $\chi_5$ be the character of the $S_5$-action on $W$.

(c) Compute the character of $\chi_5$ and show that $S_5$ operates irreducibly on $W$. So far we have four irreducible representations: $\chi_1 = 1$, $\chi_2 = \text{sgn}$, the four dimensional character $\chi_3$ and the 5-dimensional character $\chi_5$.

(d) Show that $\chi_4 := \text{sgn} \cdot \chi_3$ and $\chi_6 := \text{sgn} \cdot \chi_5$ are irreducible characters of $S_4$, different from the above four characters.

(e) Compute the seventh irreducible character $\chi_7$ of $S_5$ and complete the character table of $S_5$.

(f) Compute the character of the representation of $S_5$ on the second exterior product $\Lambda^2(U)$ of $U$ and show that it is an irreducible character. (So we have found the irreducible representation of $S_5$ with character $\chi_7$.)

5. The alternating group $A_5$ has 60 elements and 5 conjugacy classes. The elements $e$, (12)(34), (123), (12345) and (12354) are representatives of the five conjugacy classes, whose cardinalities are 1, 15, 20, 12 and 12 respectively. Note that the square of (12345) is conjugate to (12354) in $A_5$ while the inverse of (12345) is conjugate to (12345). One can obtain the character table of $A_5$ from the character table of $S_5$, worked out in problem 4.

(a) Show that the restriction to $A_5$ of the two 4-dimensional irreducible characters $\chi_3$ and $\chi_4$ of $S_5$ isomorphic and irreducible. This gives us a 4-dimensional irreducible character $\xi_2$ of $A_5$.

(b) Show that the restriction to $A_5$ of the two 5-dimensional irreducible characters $\chi_5$ and $\chi_6$ of $S_5$ are isomorphic and irreducible. This gives us a 5-dimensional irreducible character $\xi_3$ of $A_5$.

(c) Show that the restriction to $A_5$ of the 6-dimensional irreducible character $\chi_7$ of $S_5$ (which comes from $\Lambda^2(U)$ in the notation of HW 11) is the sum of two distinct irreducible 3-dimensional characters.

(d) Determine these two 3-dimensional character $\xi_4$ and $\xi_5$ and completes the character table of $A_5$. (Note that $\xi_4$ and $\xi_5$ can be relabeled to $\xi_5$ and $\xi_4$; i.e. they are specified only up to transposition of the indices.)

[Hint: You can compute for each even permutation $\sigma$ the 6 eigenvalues of the action of $\sigma$ on $\Lambda^2(U)$; $\xi_4(\sigma)$ is the sum of three of these eigenvalues, while $\xi_5(\sigma)$ is the sum of the other three eigenvalues. Use these and other constraints for the character table to determine $\xi_4$ and $\xi_5$.]