1. Shatz/Gallier Problem 137.

2. Let $G$ be a group and let $H \leq G$ be a subgroup of $G$ of finite index. Let $M$ be a left $G$-module. In class we defined the corestriction map

$$\text{Cor}_{G/H} : H^i(H,M) \rightarrow H^i(G,M)$$

via the map

$$\alpha : M \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M, \quad m \mapsto \sum_{x \in G/H} x \otimes x^{-1} m \quad \forall m \in M$$

and we defined the restriction map

$$\text{Res}_{G/H} : H_i(G,M) \rightarrow H_i(H,M)$$

via the map

$$\beta : \text{Ind}_H^G(M) \rightarrow M, \quad f \mapsto \sum_{s \in G/H} s \cdot f(x^{-1}) \quad \forall f \in \text{Ind}_H^G(M) \rightarrow M$$

We have a natural identification $H_2(G,\mathbb{Z}) \cong G/[G,G] = G^{ab}$. The restriction map

$$G^{ab} = H_2(G, \mathbb{Z}) \rightarrow H_2(H, \mathbb{Z}) = H^{ab}$$

is traditionally called the transfer map and denoted by $\text{Ver}_{G/H}$. Verify that the transfer map is equal to the map explicitly defined as follows: Choose a system of representatives $(t_i)_{i \in G/H}$ for $G/H$. For every element $s \in G$, there exists a unique element $h(s,i) \in H$ such that

$$s \cdot t_i = t_{s \cdot i} \cdot h(s,i).$$

The image $\text{Ver}_{G/H}([s])$ of the element $[s] \in G^{ab}$ represented by $s$ is the class of $\prod_{i \in G/H} h(s,i)$.

3. For any prime number $p$, the finite group $\text{GL}_2(\mathbb{F}_p)$ operates on the left of $\mathbb{F}_p^2$ via matrix multiplication.

(a) Compute $H^i(\text{GL}_2(\mathbb{F}_p), \mathbb{F}_p^2)$ for $p = 2, 3$ for as many degrees as you can.

(b) Can you generalize your result to other primes?

4. Let $\kappa$ be a finite field. Determine the Galois cohomology groups $H^i(\text{Gal}(\kappa^{\text{alg}}/\kappa), (\kappa^{\text{alg}})^{\times})$.

5. Compute the cohomology groups $H^i(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z})$ for as many degrees as you can, where the $\mathbb{Z}$ stands for the group of all integers with trivial action by $\mathbb{Z} \times \mathbb{Z}$.

6. Let $G$ be a finite group. Let $f : E \rightarrow G$ be a surjection of groups such that $A := \text{Ker}(f)$ is an abelian group, and regard $A$ as a left $G$-module via conjugation. Let $\alpha \in H^2(G,A)$ be the cohomology class corresponding to the extension

$$1 \rightarrow A \rightarrow E \xrightarrow{f} G \rightarrow 1$$

of groups. Let $(u_\sigma)_{\sigma \in G}$ be a set-theoretic lifting of $G$ to $E$, and $(a_{\sigma, \tau})_{\sigma, \tau \in G}$ be the 2-cocycle with values in $A$ representing $\alpha$ given by

$$u_\sigma \cdot u_\tau = a_{\sigma, \tau} \cdot u_{\sigma \cdot \tau}.$$
(a) Show that $\text{Ver}_{E/A}(\{a\}) = \prod_{\sigma \in G} \sigma \cdot a$ for all $a \in A$.
(b) Show that $\text{Ver}_{E/A}(\{u_\tau\}) = \prod_{\sigma \in G} a_{\sigma, \tau}$ for every $\tau \in G$.
(c) The cup product induces a map

$$\alpha \cup : G^{ab} = H^{-2}(G, \mathbb{Z}) \to \hat{H}^0(G, A) = A^G/N_G(A).$$

For an element $\sigma \in G$, let $u \in E$ be a lift of $\sigma$ in $G$. Show that the element

$$\alpha \cup [\sigma] \in \hat{H}^0(G, A)$$

is the class of $\text{Ver}_{E/A}(\{u\})$.

7. (a) If $G$ is a free group and $M$ is a left $G$-module, show that $H^i(G, M) = (0)$ for all $i \geq 0$.
(b) Let $N$ be a left $\text{PSL}_2$-module. Show that $H^i(\text{PSL}_2, N)$ is killed by 6 for all $i \geq 2$. 

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