1. Give an example of a ring \( R \), a projective \( R \)-module \( M \) and an injective \( R \)-module \( N \) such that both annihilator ideals
\[
\text{ann}_R(M) := \{ a \in R \mid a \cdot M = (0) \} \quad \text{and} \quad \text{ann}_R(N) := \{ a \in R \mid a \cdot N = (0) \}
\]
are non-zero.

2. Give an example of a ring \( R \) such that its opposite ring \( R^{opp} \) is not isomorphic to \( R \) (as rings).

3. (a) Do part 1 of problem 42 of Gallier/Shatz without assuming that the ring \( R \) is commutative.
   (b) Do parts 2 and 3 of problem 42 Gallier/Shatz, where \( R \) is assumed to be commutative.

4. Let \( R = \mathbb{F}_p[x,y]/(x^p,y^p) \), where \( p \) is a prime number.
   (a) Is \( R \) an injective module over \( R \)?
   (b) Find an injective envelope and a projective envelope of the \( R \)-module \( R/(x,y)R \).
[Recall that an injective envelope of \( R/(x,y)R \) is an \( R \)-linear essential injection \( R/(x,y)R \to I \) where \( I \) is an injective \( R \)-module; similarly for the projective envelope.]

5. Let \( \mathbb{H} \) be the ring of Hamiltonian quaternions, consisting of all formal \( \mathbb{R} \)-linear combinations of \( 1, i, j, k \), such that \( i^2 = j^2 = k^2 = -1 \) and
\[
i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k.
\]
The conjugation \( t: \mathbb{H} \to \mathbb{H} \) sends an element \( a + bi + c j + dk \in \mathbb{H} \) to \( a - bi - c j - dk \), where \( a, b, c, d \) are real numbers. The norm of an element \( z = a + bi + c j + dk \in \mathbb{H} \) is defined to be
\[
\text{Nm}(z) := z \cdot t(z) = t(z) \cdot z = a^2 + b^2 + c^2 + d^2
\]
(a) Show that \( \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \) is isomorphic to \( M_2(\mathbb{C}) \) as a ring and exhibit an explicit isomorphism.
(b) Let \( \mathbb{H}_1^\times \) be the subgroup of the \( \mathbb{H}^\times \) consisting of all elements of \( \mathbb{H}^\times \) of norm 1. Use the isomorphism you found in (a) to give an explicit subgroup of \( M_2(\mathbb{C}) \) isomorphic to \( \mathbb{H}_1^\times \).
(c) The three-dimensional \( \mathbb{R} \)-vector subspace \( W := \mathbb{R} \cdot i + \mathbb{R} \cdot j + \mathbb{R} \cdot k \) of \( \mathbb{H} \) is stable under conjugation by group \( \mathbb{H}^\times \) of units in \( \mathbb{H} \), so we have a group homomorphism
\[
\alpha: \mathbb{H}^\times \to \text{GL}_{\mathbb{R}}(W).
\]
Determine the kernel and the image of \( \alpha \); in particular give an alternative/implicit description of the image of \( \alpha \) via the identification of \( \text{GL}_{\mathbb{R}}(W) \) with \( \text{GL}_3(\mathbb{R}) \) using the basis \( i, j, k \) of \( W \).
[Note: Part (c) is independent of part (a) but of related interest.]