1. Let $R$ be a ring and let $M$ be a finitely presentable left $R$-module, i.e. there exists an exact sequence

$$R^n \rightarrow R^m \rightarrow M \rightarrow 0$$

of left $R$-modules. Suppose that $\alpha : R^N \rightarrow M$ is a surjective $R$-module homomorphism. Is $\text{Ker}(\alpha)$ a finitely generated left $R$-module? Either prove the statement or give a counter-example.

2. Do Gallier-Shatz problem 84, part 1.

3. Let $M, N$ be modules over a commutative ring $R$. Let $S$ be a multiplicatively closed subset of $R - \{0\}$ containing 1. We have a natural $R_S$-linear homomorphism

$$h : \text{Hom}_R(M, N) \rightarrow \text{Hom}_{R_S}(M_S, N_S).$$

(a) If $M$ is finitely generated, then $h$ is injective.

(b) If $M$ is finitely presentable then $h$ is bijective.

(c) Does the statement (a) hold without the assumption that $M$ is finitely generated? Either prove the statement or give a counter-example.

(d) Does the statement (b) hold for all finitely generated $R$-modules $M$? Either prove the statement or give a counter-example.

4. Let $R$ be a ring and let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be a short exact sequence of left $R$-modules. Assume that $M$ and $M''$ are finitely presentable left $R$-modules.

(a) Is $M'$ finitely generated? Either give a proof or a counter-example.

(b) Suppose that $M'$ is finitely generated. Is $M'$ finitely presentable? Either give a proof or a counter-example.

5. Let $R$ be a commutative ring. Prove that $\text{Spec}(R)$ is a connected topological space if and only if the only idempotents in $R$ are 0 and 1.

[An element $x \in R$ is an idempotent if and only if $x^2 = x$.]

6. Let $R$ be a noetherian commutative ring. Let $Y$ be a closed subset of $\text{Spec}(R)$. Assume that

- $\text{Spec}(R)$ is connected, and
- $\text{Spec}(R_p) - \{[p]\}$ is connected for every point $[p] \in Y$.

(Here $\text{Spec}(R_p) - \{[p]\}$ is given the induced topology.)

Prove that $\text{Spec}(R) - Y$ is connected.

**Plan for 11/10/2010–11/12/2010.**

- integral extension of rings.
- prime ideal chains in integral ring extensions: lying over, going up and going down.
- noether normalization theorem.