The Forster/Swan theorem

§1. Notation and definition

In this note $R$ is a commutative ring with 1, $Y := \text{Spec}(R)$ is the spectrum of $R$, $X := J(R)$ is the $J$-spectrum of $R$.

(1.1) **Definition** Let $M$ be a finitely generated $R$-module. For every prime ideal $p \in Y$, define

$$\mu(p, M) = \dim_{\kappa(p)}(M_p/pM_p)$$

to be the minimum number of generators for the $R_p$-module $M_p$. It is easy to see that $p \mapsto \mu(p, M)$ is an upper semi-continuous function on $\text{Spec}(R)$; i.e. the subset

$$Y_{n, M} := \{ p \in \text{Spec}(R) \mid \mu(p, M) \geq n \} \subseteq Y$$

is closed for every $n \in \mathbb{N}$. Let

$$X_{n, M} := X \cap Y_{n, M} \quad \text{for all } n \in \mathbb{N}.$$

(1.2) **Lemma** Let $R$ be any commutative ring, let $M$ be a finitely generated $R$-module, and let $S$ be a finite set of prime ideals of $R$ such that $M_p \neq 0$ for every $p \in S$. There exists an element $m \in M$ such that the image of $m$ in $M_p/pM_p$ is non-zero for all $p \in S$. In other words

$$\mu(p, M/R \cdot m) = \mu(p, M) - 1 \quad \forall p \in S.$$

**Proof.** Induction on $s = \text{card}(S)$. The statement obviously holds for $s = 0$. Suppose that the statement holds for $s - 1$, $s \geq 1$. Pick a minimal element in $S$ and call it $p_s$. Let $\{p_1, \ldots, p_{s-1}\}$ be the rest of the elements in $S$. Then $p_1 \cdots p_{s-1} \not\subseteq p_s$. Pick an element $x_s \in p_1 \cdots p_{s-1} - p_s$. According to the induction hypothesis, there is an element $m_1 \in M$ such that the image of $m_1$ in $M_{p_i}/p_iM_{p_i}$ is non-zero for $i = 1, \ldots, s-1$. Pick an element $m_2 \in M$ whose image in $M_{p_s}/p_sM_{p_s}$ is non-zero. The element

$$m = \begin{cases} m_1 & \text{if the image of } m_1 \text{ in } M_{p_i}/p_iM_{p_i} \text{ is non-zero} \\ m_1 + x_s m_2 & \text{otherwise} \end{cases}$$

of $M$ has the required property. □

(1.3) In the rest of this note we assume that $X$ is a noetherian topological space and $\dim(X) < \infty$.

For any finitely generated $R$-module $M$, define a function $b(p, M)$ on $X$ by

$$b(p, M) = \begin{cases} 0 & \text{if } M_p = 0 \\ \mu(p, M) + \dim(V(p) \cap X) & \text{otherwise.} \end{cases}$$

Here $\dim(V(p) \cap X)$ is the combinatorial dimension of the topological space $V(p) \cap X$, which is equal to the combinatorial dimension of $V(p) \cap \text{Max}(R)$. Clearly the closed subset $X_{n, M}$ of $X$ is empty for $n \gg 0$ and the function $p \mapsto b(p, M)$ is bounded on $X$, because $\dim(X)$ is assumed to be finite.
§ 2. The theorem

(2.1) Theorem. Let \( R \) be a commutative ring such that the \( X = J(R) \) is noetherian as above. Let \( M \) be a finitely generated \( R \)-module. Let \( k \) be a natural number such that \( k \geq \max_{p \in X} b(p, M) \). Then \( M \) can be generated over \( R \) by \( k \) elements.

Proof. Induction on \( k \); the case \( k = 0 \) is obvious. Suppose that the statement holds for \( k - 1, k \geq 1 \). Let \( S \) be the subset of \( X \) consisting of all prime ideals \( p \in X \) such that \( b(p, M) = k \).

Consider an element \( q \in S \), and let \( n = \mu(q, M) \). Let \( Z \) be the closure of \( \{q\} \). Then \( Z \) is an irreducible component of \( X_{n,M} \); otherwise there exists an element \( p \in X_{n,M} \) such that \( p \subseteq q \) and

\[
b(p, M) = \mu(p, M) + \dim(V(p \cap X)) \geq n + \dim(V(p \cap X)) > n + \dim(V(q \cap X)) = k,
\]

a contradiction. So \( S \) is a finite set, contained in the finite set consisting of all generic points of irreducible subsets of the finitely many non-empty closed subsets \( X_{n,M} \subset X \).

By Lemma 1.2, there exists an element \( m \in M \) such that \( \mu(q, M/R \cdot m) = \mu(q, M) - 1 \) for all \( q \in S \). Then \( b(p, M/R \cdot m) \leq k - 1 \) for all \( p \in X \), so \( M/R \cdot m \) can be generated by \( k - 1 \) elements by the induction hypothesis. \( \square \)