Notes on semisimple algebras

§1. Semisimple rings

(1.1) Definition A ring R with 1 is *semisimple*, or left semisimple to be precise, if the free left R-module underlying R is a sum of simple R-module.

(1.2) Definition A ring R with 1 is *simple*, or left simple to be precise, if R is semisimple and any two simple left ideals (i.e. any two simple left submodules of R) are isomorphic.

(1.3) Proposition *A ring R is semisimple if and only if there exists a ring S and a semisimple S-module M of finite length such that $R \cong \mathrm{End}_S(M)$*

(1.4) Corollary *Every semisimple ring is Artinian.*

(1.5) Proposition *Let R be a semisimple ring. Then R is isomorphic to a finite direct product $\prod_{i=1}^{s} R_i$, where each R_i is a simple ring.*

(1.6) Proposition *Let R be a simple ring. Then there exists a division ring D and a positive integer n such that $R \cong \mathrm{M}_n(D)$.*

(1.7) Definition Let R be a ring with 1. Define the *radical* of R to be the intersection of all maximal left ideals of R. The above definitions uses left R-modules. When we want to emphasize that, we say that $\mathfrak{n}$ is the *left radical* of R.

(1.8) Proposition *The radical of a semisimple ring is zero.*

(1.9) Proposition *Let R be a simple ring. Then R has no non-trivial two-sided ideals, and its radical is zero.*

(1.10) Proposition *Let R be an Artinian ring whose radical is zero. Then R is semisimple. In particular, if R has no non-trivial two-sided ideal, then R is simple.*

(1.11) Remark In non-commutative ring theory, the standard definition for a ring to be semisimple is that its radical is zero. This definition is *different* from Definition 1.1, For instance, $\mathbb{Z}$ is not a semisimple ring in the sense of Def. 1.1, while the radical of $\mathbb{Z}$ is zero. In fact the converse of Prop. 1.10 holds; see Cor. 1.4 below.

(1.12) Exercise. Let R be a ring with 1. Let $\mathfrak{n}$ be the radical of R

(i) Show that there exists a maximal left ideal in R. Deduce that the radical of R is a proper left ideal of R. (Hint: Use Zorn's Lemma.)

(ii) Show that $\mathfrak{n} \cdot M = (0)$ for every simple left R-module M. (Hint: Show that for every $0 \neq x \in M$, the set of all elements $y \in R$ such that $y \cdot x = 0$ is a maximal left ideal of R.)

(iv) Suppose that I is a left ideal of R such that $I \cdot M = (0)$ for every simple left R-module M. Prove that $I \subseteq \mathfrak{n}$.

(v) Show that $\mathfrak{n}$ is a two-sided ideal of R. (Hint: Use (iv).)

(vi) Let I be a left ideal of R such that $I^n = (0)$ for some positive integer n. Show that $I \subseteq \mathfrak{n}$.

(vi) Show that the radical of $R/\mathfrak{n}$ is zero.

(1.13) Exercise. Let R be a ring with 1 and let $\mathfrak{n}$ be the (left) radical of R.

(i) Let $x \in \mathfrak{n}$. Show that $R \cdot (1 + x) = R$, i.e. there exists an element $z \in R$ such that $z \cdot (1 + x) = 1$.

(ii) Suppose that J is a left ideal of R such that $R \cdot (1 + x) = R$ for every $x \in J$. Show that $J \subseteq \mathfrak{n}$. (Hint: If not, then there exists a maximal left ideal $\mathfrak{m}$ of R such that $J + \mathfrak{m} \ni 1$.)

(iii) Let $x \in \mathfrak{n}$, and let z be an element of R such that $z \cdot (1 + x) = 1$. Show that $z - 1 \in \mathfrak{n}$. Conclude that $1 + \mathfrak{n} \subset R^\times$.

(iv) Show that the $\mathfrak{n}$ is equal to the right radical of R. (Hint: Use the analogue of (i)–(iii) for the right radical.)

§2. Simple algebras

(2.1) Proposition *Let K be a field. Let A be a central simple algebra over K, and let B be simple K-algebra. Then $A \otimes_K B$ is a simple K-algebra. Moreover $\mathrm{Z}(A \otimes_K B) = \mathrm{Z}(B)$, i.e. every element of the center of $A \otimes_K B$ has the form $1 \otimes b$ for a unique element $b \in \mathrm{Z}(B)$. In particular, $A \otimes_K B$ is a central simple algebra over K if both A and B are.*

Proof. We assume for simplicity of exposition that $\dim_K(B) < \infty$; the proof works for the infinite dimensional case as well. Let $b_1, \ldots, b_r$ be a K-basis of B. Define the *length* of an element $x = \sum_{i=1}^r a_i \otimes b_i \in A \otimes B$, $a_i \in A$ for $i = 1, \ldots, r$, to be $\mathrm{Card}\{\, i \mid a_i \neq 0 \,\}$.

Let I be a non-zero ideal in $A \otimes_K B$. Let x be a non-zero element of I of minimal length. After relabelling the b_i's, we may and do assume that x has the form

$$x = a_1 \otimes b_1 + \sum_{i=2}^{s} a_i \otimes b_i \,,$$

and $a_1, \ldots, a_s$ are all non-zero. Since $a_1 \neq 0$ and A is simple, there exist elements $u_1, u_2, \ldots, u_h$ and $v_1, v_2, \ldots, v_h$ in A such that $\sum_{j=1}^h u_j \, a_1 \, v_j = 1$. Consider the element

$$y = \sum_{j=1}^{h} (u_j \otimes 1) \cdot x \cdot (v_j \otimes 1) \in I \,.$$

We have

$$y = 1 \otimes b_1 + \sum_{i=2}^{s} a_i' \otimes b_i$$

where $a_i' = \sum_{j=1}^h u_j \cdot a_i \cdot v_j$ for $j = 2, \ldots, s$. Clearly $y \neq 0$ and its is at most s. So y has length s and $a_i' \neq 0$ for $i = 2, \ldots, s$. Consider the element $[a \otimes 1, y] \in I$ with $a \in A$, whose length is strictly less than s. Therefore $[a \otimes 1, y] = 0$ for all $a \in A$, i.e. $[a, a_i'] = 0$ for all $a \in A$ and all $i = 2, \ldots, s$. In other words, $a_i' \in K$ for all $i = 2, \ldots, s$. Write $a_i' = \lambda_i \in K$, and $y = 1 \otimes b \in I$, where $b = b_1 + \lambda_2 b_2 + \cdots \lambda_s b_s \in B$, $b \neq 0$. So $1 \otimes BbB \subseteq I$. Since B is simple, we have $BbB = B$ and hence $I = A \otimes_K B$. We have shown that $A \otimes_K B$ is simple.

Next we prove that $\mathrm{Z}(A \otimes_K B) = K$. Let $x = \sum_{i=1}^r a_i \otimes b_i$ be any element of $\mathrm{Z}(A \otimes_K B)$, with $a_1, \ldots, a_r \in A$. We have

$$0 = [a \otimes 1, x] = \sum_{i=1}^r [a, a_i] \otimes b_i$$

for all $a \in A$. Hence $a_i \in \mathrm{Z}(A) = K$ for each $i = 1, \ldots, r$, and $x = 1 \otimes b$ for some $b \in B$. The condition that $0 = [1 \otimes y, x]$ for all $y \in B$ implies that $b \in \mathrm{Z}(B)$ and hence $x \in 1 \otimes \mathrm{Z}(B)$. ∎

(2.2) Corollary *Let A be a finite dimensional algebra over a field K, and let $n = \dim_K(A)$. If A is a central simple algebra over K, then*

$$A \otimes_K A^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{End}_K(A) \cong \mathrm{M}_n(K)\,.$$

Conversely, if $A \otimes_K A^{\mathrm{opp}} \twoheadrightarrow \mathrm{End}_K(A)$, then A is a central simple algebra over K. ∎

Proof. Suppose that A is a central simple algebra over K. By Prop. 2.1, $A \otimes_K A^{\mathrm{opp}}$ is a central simple algebra over K. Consider the map

$$\alpha : A \otimes_K A^{\mathrm{opp}} \to \mathrm{End}_K(A)$$

which sends $x \otimes y$ to the element $u \mapsto xuy \in \mathrm{End}_K(A)$. The source of α is simple by Prop. 2.1, so α is injective because it is clearly non-trivial. Hence it is an isomorphism because the source and the target have the same dimension over K.

Conversely, suppose that $A \otimes_K A^{\mathrm{opp}} \twoheadrightarrow \mathrm{End}_K(A)$ and I is a proper ideal of A. Then the image of $I \otimes A^{\mathrm{opp}}$ in $\mathrm{End}_K(A)$ is an ideal of $\mathrm{End}_K(A)$ which does not contain Id_A. so A is a simple K-algebra. Let $L := \mathrm{Z}(A)$, then the image of the canonical map $A \otimes_K A^{\mathrm{opp}}$ in $\mathrm{End}_K(A)$ lies in the subalgebra $\mathrm{End}_L(A)$, hence $L = K$. ∎

(2.3) Lemma *Let D be a finite dimensional central division algebra over an algebraically closed field K. Then $D = K$.* ∎

(2.4) Corollary *The dimension of any central simple algebra over a field is a perfect square.*

(2.5) Lemma *Let A be a finite dimensional central simple algebra over a field K. Let $F \subset A$ be an overfield of K contained in A. Then $[F : K] \mid [A : K]^{1/2}$. In particular if $[F : K]^2 = [A : K]$, then F is a maximal subfield of A.*

Proof. Write $[A : K] = n^2$, $[F : K] = d$. Multiplication on the left defines an embedding $A \otimes_K F \hookrightarrow \mathrm{End}_F(A)$. By Lemma 3.1, $n^2 = [A \otimes_K : F]$ divides $[\mathrm{End}_F(A) : F] = (n^2/d)^2$, i.e. $d^2 \mid n^2$. So d divides n. ∎

(2.6) Lemma *Let A be a finite dimensional central simple algebra over a field K. If F is a subfield of A containing K, and $[F : K]^2 = [A : K]$, then F is a maximal subfield of K and $A \otimes_K F \cong \mathrm{M}_n(F)$, where $n = [A : K]^{1/2}$.*

Proof. We have seen in Lemma 2.5 that F is a maximal subfield of A. Consider the natural map $\alpha : A \otimes_K F \to \mathrm{End}_K(A)$, which is injective because $A \otimes_K F$ is simple and α is non-trivial. Since the dimension of the source and the target of α are both equal to n^2, α is an isomorphism. ∎

(2.7) Proposition *Let D be a non-commutative central division algebra over a field K, There exists an element $u \notin K$ which is separable over K.*

Proof. Suppose that every element $u \notin K$ is purely inseparable over K. Clearly K is infinite. The assumption implies that the minimial polynomial of every element of D has the form $T^{p^i} - a$ for some $i \in \mathbb{N}$ and some $a \in K$. Moreover $p^i \leq \dim_K(D)^{1/2}$. So there exists an integer N such that $x^{p^N} \in K$ for all $x \in D$. Therefore $[x^{p^N}, y] = 0$ for all $x, y \in D$.

Let $\underline{D}$ be the affine K-scheme such that $\underline{D}(L) = D \otimes_K L$ for every extension field L/K. There is a K-morphism

$$f : \underline{D} \times_{\mathrm{Spec}(K)} \underline{D} \to \underline{D}$$

such that $f(x, y) = [x^{p^N}, y]$ for all extension field L/K and all $x, y \in \underline{D}(L)$. We know that this morphism is zero on the dense subset $\underline{D}(K) \times \underline{D}(K)$, hence f is the zero morphism. The last statement is impossible, for $\underline{D}(K^{\mathrm{alg}}) \cong \mathrm{M}_r(L^{\mathrm{alg}})$ with $r = \dim_K(D)^{1/2} > 0$ and the equality $[x^{p^N}, y] = 0$ for all $x, y \in \mathrm{M}_r(L^{\mathrm{alg}})$ is absurd. $\square$

(2.8) Theorem (Noether-Skolem) *Let B be a finite dimensional central simple algebra over a field K. Let A_1, A_2 be simple K-subalgebras of B. Let $\phi : A_1 \xrightarrow{\sim} A_2$ be a K-linear isomorphism of K-algebras. Then there exists an element $x \in B^\times$ such that $\phi(y) = x^{-1}yx$ for all $y \in A_1$.*

Proof. Consider the simple K-algebra $R := B \otimes_K A_1^{\mathrm{opp}}$, and two R-module structures on the K-vector space V underlying B: an element $u \otimes a$ with $u \in B$ and $a \in A_1^{\mathrm{opp}}$ operates either as $b \mapsto uba$ for all $b \in V$, or as $b \mapsto ub\phi(a)$ for all $b \in V$. Hence there exists a $\psi \in \mathrm{GL}_K(V)$ such that

$$\psi(uba) = u\psi(b)\phi(a)$$

for all $u, b \in B$ and all $a \in A_1$. One checks easily that $\psi(1) \in B^\times$: if $u \in B$ and $u \cdot \psi(1) = 0$, then $\psi(u) = 0$, hence $u = 0$. Then $\phi(a) = \psi(1)^{-1} \cdot a \cdot \psi(1)$ for every $a \in A_1$. $\square$

(2.9) Theorem *Let B be a K-algebra and let A be a finite dimensional central simple K-subalgebra of B. Then the natural homomorphism $\alpha : A \otimes_K \mathrm{Z}_B(A) \to B$ is an isomorphism.*

Proof. Passing from K to K^{alg}, we may and do assume that $A \cong \mathrm{M}_n(K)$, and we fix an isomorphism $A \xrightarrow{\sim} \mathrm{M}_n(K)$.

First we show that α is surjective. Given an element $b \in B$, define elements $b_{ij} \in B$ for $1 \leq i, j \leq n$ by

$$b_{ij} := \sum_{k=1}^{n} e_{ki}\, b\, e_{jk}\,,$$

where $e_{ki} \in \mathrm{M}_n(K)$ is the $n \times n$ matrix whose (k, i)-entry is equal to 1 and all other entries equal to 0. One checks that each b_{ij} commutes with all elements of $A = \mathrm{M}_n(K)$. The following computation

$$\sum_{i,j=1}^{n} b_{ij}\, e_{ij} = \sum_{i,j,k} e_{ki}\, b\, e_{jk}\, e_{ij} = \sum_{i,j} e_{ii}\, b\, e_{jj} = b$$

shows that α is surjective.

Suppose that $0 = \sum_{i,j=1}^{n} b_{ij}e_{ij}$, $b_{ij} \in \mathrm{Z}_B(A)$ for all $1 \le i,j \le n$. Then

$$0 = \sum_{k=1}^{n} e_{kl} \left(\sum_{i,j} b_{ij}e_{ij} \right) e_{mk} = \sum_{k=1}^{n} b_{lm}e_{kk} = b_{lm}$$

for all $0 \le l,m \le n$. Hence α is injective. ∎

(2.10) Theorem *Let B be a finite dimensional central simple algebra over a field K, and let A be a simple K-subalgebra of B. Then $\mathrm{Z}_B(A)$ is simple, and $\mathrm{Z}_B(\mathrm{Z}_B(A)) = A$.*

Proof. Let $C = \mathrm{End}_K(A) \cong \mathrm{M}_n(K)$, where $n = [A : K]$. Inside the central simple K-algebra $B \otimes_K C$ we have two simple K-subalgebras, $A \otimes_K K$ and $K \otimes_K A$. Here the right factor of $K \otimes_K A$ is the image of A in $C = \mathrm{End}_K(A)$ under left multiplication. Clearly these two simple K-subalgebras of $B \otimes_K C$ are isomorphic, since both are isomorphic to A as a K-algebra. By Noether-Skolem, these two subalgebras are conjugate in $B \otimes_K C$ by a suitable element of $(B \otimes_K C)^\times$, therefore their centralizers (resp. double centralizers) in $B \otimes_K C$ are conjugate, hence isomorphic.

Let's compute the centralizers first:

$$\mathrm{Z}_{B\otimes_K C}(A \otimes_K K) = \mathrm{Z}_B(A) \otimes_K C\,,$$

while

$$\mathrm{Z}_{B\otimes_K C}(K \otimes_K A) = B \otimes_K A^{\mathrm{opp}}\,.$$

Since $B \otimes_K A^{\mathrm{opp}}$ is central simple over K, so is $\mathrm{Z}_B(A) \otimes_K C$. Hence $\mathrm{Z}_B(A)$ is simple.

We compute the double centralizers:

$$\mathrm{Z}_{B\otimes_K C}(\mathrm{Z}_{B\otimes_K C}(A \otimes_K K)) = \mathrm{Z}_{B\otimes_K C}(\mathrm{Z}_B(A) \otimes_K C) = \mathrm{Z}_B(\mathrm{Z}_B(A) \otimes_K K\,,$$

while

$$\mathrm{Z}_{B\otimes_K C}(\mathrm{Z}_{B\otimes_K C}(K \otimes_K A)) = \mathrm{Z}_{B\otimes_K C}(B \otimes_K A^{\mathrm{opp}}) = K \otimes_K A$$

So $\mathrm{Z}_B(\mathrm{Z}_B(A))$ is isomorphic to A as K-algebras. Since $A \subseteq \mathrm{Z}_B(\mathrm{Z}_B(A))$, the inclusion is an equality. ∎

§3. Some invariants

(3.1) Lemma *Let K be a field and let A be a finite dimensional simple K-algebra. Let M be an (A, A)-bimodule. Then M is free as a left A-module.*

Proof. We have $A \cong \mathrm{M}_n(D)$ for some division K-algebra D. To say that M is free means that $\mathrm{length}_A(M) \equiv 0 \pmod{n}$. Let A_s be the left A-module underlying A. Because A_s is isomorphic direct sum of n copies of the irreducible left A-module $N := \mathrm{M}_{n\times 1}(D)$, we have

$$M \cong M \otimes_A A_s \cong (M \otimes_A N)^{\oplus n}\,.$$

So $\mathrm{length}_A(M) \equiv 0 \pmod{n}$. ∎

(3.2) Definition Let K be a field, B be a K-algebra, and let A be a finite dimensional simple K-subalgebra of B. Then B is a free left A-module by Lemma 3.1. We define the *rank* of B over A, denoted $[B:A]$, to be the rank of B as a free left A-module. Clearly $[B:A] = \dim_K(B)/\dim_K(A)$ if $\dim_K(A) < \infty$.

(3.3) Definition Let K be a field. Let B be a finite dimensional simple K-algebra, and let A be a simple K-subalgebra of B. Let N be a left simple B-module, and let M be a left simple A-module.

(i) Define $i(B,A) := \mathrm{length}_B(B \otimes_A M)$, called the *index* of A in B.

(ii) Define $h(B,A) := \mathrm{length}_A(N)$, called the *height* of B over A.

Here $[B:A]$ denotes the A-rank of B_s, where B_s is the free left A-module underlying B.

(3.4) Lemma *Notation as in Def.* 3.3.

(i) $\mathrm{length}_B(B \otimes_A U) = i(B,A)\, \mathrm{length}_A(U)$ *for every left A-module U.*

(ii) $\mathrm{length}_A(V) = h(B,A) \cdot \mathrm{length}_B(V)$ *for every left B-module V.*

(iii) $\mathrm{length}_B(B_s) = i(B,A) \cdot \mathrm{length}_A(A_s)$.

(iv) $\mathrm{length}_A(B \otimes_A U) = [B:A] \cdot \mathrm{length}_A(U)$ *for every left A-module U.*

(v) $[B:A] = h(B,A) \cdot i(B,A)$

Proof. Statements (i), (ii) follow immediately from the definition. The statement (iii) follows from (i) and the fact that $B_s \cong B \otimes_A A_s$. The statement (iv) holds for $U = A_s$ from the definition of $[B:A]$, hence it hold for all left A-modules U. To show (v), we apply (iv) to a simple A-module M and get

$$[B:A] = \mathrm{length}_A(B \otimes_A M) = h(B,A)\, \mathrm{length}_B(B \otimes_A M) = h(B,A)\, i(B,A)\,.$$

Another proof of (iv) is to use the A-module A_s instead of a simple A-module M:

$$[B:A]\, \mathrm{length}_A(A_s) = \mathrm{length}_A(B_s) = \mathrm{length}_B(B_s)\, h(B,A) = h(B,A)\, i(B,A)\, \mathrm{length}_A(A_s)\,.$$

The last equality follows from (iii). □

(3.5) Lemma *Let $A \subset B \subset C$ be inclusion of simple algebras over a field K. Then $i(C,A) = i(C,B) \cdot i(B,A)$, $h(C,A) = h(C,B) \cdot h(B,A)$, and $[C:A] = [C:B] \cdot [B:A]$.* □

(3.6) Lemma *Let K be an algebraically closed field. Let B be a finite dimensional simple K-algebra, and let A be a semisimple K-subalgebra of B. Let M be a simple A-module, and let N be a simple B-module.*

(i) *N contains M as a left A-module.*

(ii) *The following equalities hold.*

$$\begin{aligned}\dim_K(\mathrm{Hom}_B(B \otimes_A M, N)) &= \dim_K(\mathrm{Hom}_A(M,N)) = \dim_K(\mathrm{Hom}_A(N,M)) \\ &= \dim_K(\mathrm{Hom}_B(N, \mathrm{Hom}_A(B,M)))\end{aligned}$$

(iii) *Assume in addition that A is simple. Then $i(B,A) = h(B,A)$.*

PROOF. Statements (i), (ii) are easy and left as exercises. The statement (iii) follows from the first equality in (ii). □

(3.7) Lemma *Let A be a simple algebra over a field K. Let M be a non-trivial finitely generated left A-module, and let $A' := \mathrm{End}_A(M)$. Then $\mathrm{length}_A(M) = \mathrm{length}_{A'}(A'_s)$, where A'_s is the left A'_s-module underlying A'.*

PROOF. Write $M \cong U^n$, where U is a simple A-module. Then $A' \cong \mathrm{M}_n(D)$, where $D := \mathrm{End}_A(U)$ is a division algebra. So $\mathrm{length}_{A'}(A'_s) = n = \mathrm{length}_A(M)$. □

(3.8) Proposition *Let K be a field, B be a finite dimensional simple K-algebra, and let A be a simple K-subalgebra of B. Let N be a non-trivial B-module. Then*

(i) *$A' := \mathrm{End}_A(N)$ is a simple K-algebra, and $B' := \mathrm{End}_B(N)$ is a simple K-subalgebra of A'.*

(ii) *$i(A', B') = h(B, A)$, and $h(A', B') = i(B, A)$.*

PROOF. The statement (i) is easy and omitted. To prove (ii), we have

$$\mathrm{length}_A(N) = \mathrm{length}_{A'}(A'_s) = i(A', B')\, \mathrm{length}_{B'}(B'_s)\,,$$

where the first equality follows from Lemma 3.7 and the second equality follows from Lemma 3.4 (iii). We also have

$$\mathrm{length}_A(N) = h(B, A)\, \mathrm{length}_B(N) = h(B, A)\, \mathrm{length}_{B'}(B'_s)$$

where the last equality follows from Lemma 3.7. So we get $i(A', B') = h(B, A)$. Replacing (B, A) by (A', B'), we get $i(B, A) = h(A', B')$. □

(3.9) Extending the method in , we can express the invariants $i(B, A)$ and $h(B, A)$ somewhat more explicitly in terms of the basic invariants of B and A. Write

$$A \cong \mathrm{M}_m(D), \quad B \cong \mathrm{M}_n(E)$$

where E is a central division algebra over K, and E is a central division algebra over a finite extension field L/K. Let $d^2 = \dim_L(D)$, $e^2 = \dim_K(E)$. Let $M \cong D^{\oplus m}$, $N \cong E^{\oplus n}$, with their natural module structure over $\mathrm{M}_m(D)$ and $\mathrm{M}_n(E)$ respectively. The canonical isomorphism

$$\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N)$$

gives us the equality

$$i(B, A) \cdot e^2 = h(B, A) \cdot d^2 \cdot [L : K]\,.$$

Together with $i(B, A) \cdot h(B, A) = [B : A] = \frac{n^2\, e^2}{m^2\, d^2\, [L:K]}$, we get

$$i(B, A) = \frac{n}{m}, \qquad h(B, A) = \frac{[B : A]}{(n/m)}\,.$$

In particular, $m \mid n$, and $(n/m) \mid [B : A]$.

§4. Centralizers

(4.1) Theorem *Let K be a field. Let B be a finite dimensional central simple algebra over K. Let A be a simple K-subalgebra of B, and let $A' := \mathrm{Z}_B(A)$. Let $L := A \cap A' = \mathrm{Z}(A)$. Then the following holds.*

(i) *A' is a simple K-algebra.*

(ii) *$A := \mathrm{Z}_B(\mathrm{Z}_B(A))$.*

(iii) *$[B : A'] = [A : K]$, $[B : A] = [A' : K]$, $[B : K] = [A : K] \cdot [A' : K]$.*

(iv) *$L = \mathrm{Z}(A) = \mathrm{Z}(A')$; A and A' are linearly disjoint over L.*

(v) *If A is a central simple algebras over K, then $A \otimes_K A' \xrightarrow{\sim} B$.*

(vi) *For any non-trivial B-module N we have natural isomorphisms*

$$\mathrm{End}_B(N) \otimes_K A \xrightarrow{\sim} \mathrm{End}_{\mathrm{Z}_B(A)}(N), \qquad \mathrm{End}_B(N) \otimes_K \mathrm{Z}_B(A) \xrightarrow{\sim} \mathrm{End}_A(N)\,.$$

Remark (1) Statements (i) and (ii) of Thm. 4.1 is the content of double centralizer theorem 2.10. The proof in 2.10 uses Noether-Skolem and the fact that the double centralizer of any K-algebra A in $\mathrm{End}_K(A)$ is equal to itself. The proof in 4.1 relies on Prop. 3.8.

(2) Statement (v) of Thm. 4.1 is a special case of Thm. 2.9.

Proof. Let N be a non-trivial left B-module. Let $D := \mathrm{End}_B(N)$. We have $D \subseteq \mathrm{End}_K(N) \supseteq B$, and $\mathrm{Z}(D) = \mathrm{Z}(B) = K$. So $D \otimes_K A$ is a simple K-algebra, and we have

$$D \otimes_K A \xrightarrow{\sim} D \cdot A \subseteq \mathrm{End}_K(N) =: C$$

where $D \cdot A$ is the subalgebra of $\mathrm{End}_K(N)$ generated by D and A. We have $B \cong \mathrm{M}_n(R)$ for some central division K-algebra R. Under this identification we can take N to be $\mathrm{M}_{n\times 1}(R)$. Then $D = R^{\mathrm{opp}}$, operating naturally on $\mathrm{M}_{n\times 1}(R)$, and $\mathrm{Z}_C(D) = B$. So we know one instance of the double centralizer theorem; $\mathrm{Z}_C(\mathrm{Z}_C(B)) = B$. We will leverage this one instance to prove the general double centralizer theorem. We have

$$\mathrm{Z}_C(D \cdot A) = \mathrm{Z}_C(D) \cap \mathrm{Z}_C(A) = B \cap \mathrm{Z}_C(A) = A'\,.$$

Hence $A' = \mathrm{End}_{D\cdot A}(N)$ is simple, because $D \cdot A$ is simple. We have proved (i).

Apply Prop. 3.8 (ii) to the pair $(D \cdot A, D)$ and the $D \cdot A$-module N. We get

$$[A : K] = [D \cdot A : D] = [B : A']$$

since $\mathrm{Z}_C(D) = B$. On the other hand, we have

$$[B : A] \cdot [A : K] = [B : K] = [B : A'] \cdot [A' : K] = [A : K] \cdot [A' : K]$$

so $[B : A] = [A' : K]$. We have proved (iii).

Apply (i) and (iii) to the simple K-subalgebra $A' \subseteq B$, we see that $[A : K] = [\mathrm{Z}_B(A') : K]$, so $A = \mathrm{Z}_B(\mathrm{Z}_B(A))$ because $A \subset \mathrm{Z}_B(A')$. We have proved (ii).

Let $L := A \cap A' = \mathrm{Z}(A) \subseteq \mathrm{Z}(A') = \mathrm{Z}(A)$. The last equality follows from (ii). The tensor product $A \otimes_L A'$ is a central simple algebra over L since A and A' are central simple over L. So the canonical homomorphism $A \otimes_L A' \to B$ is an injection. We have prove (iv). The above inclusion is an equality if and only if $L = K$, because $\dim_L(B) = [L : K] \cdot [A : L] \cdot [A' : L]$.

We have seen in the proof of (i) that the centralizer of the image C of $\mathrm{End}_B(N) \otimes_K A$ in $\mathrm{End}_K(N)$ is $\mathrm{Z}_B(A)$. So C is equal to $\mathrm{End}_{\mathrm{Z}_B(A)}(N)$. We have proved the first equality in (vi). The second equality in (vi) follows. □

Remark (a) Theorem 4.1 (iii) is crucial in 4.3–4.6 below.
(b) One can also finish the proof of (vi) by dimension count, after having shown a natural injection $\mathrm{End}_B(N) \otimes_K A' \hookrightarrow \mathrm{End}_A(N)$. Let $r = \dim_K(N)$. Then

- $\dim_K(\mathrm{End}_A(N)) = r^2/\dim_K(A)$,
- $\dim_K(\mathrm{End}_B(N)) = r^2/\dim_K(B)$,
- $\dim_K(A') = \dim_K(B)/\dim_K(A)$,

all by 4.1 (iii). So $\dim_K(\mathrm{End}_B(N) \otimes_K A') = \dim_K(\mathrm{End}_A(N))$.

(4.2) Corollary *Notation as in* 4.1. *Let* $L := \mathrm{Z}(A) = \mathrm{Z}(\mathrm{Z}_B(A))$. *Then* $[A \otimes_L \mathrm{Z}_B(A)]$ *and* $[B \otimes_K L]$ *are equal as elements of* $\mathrm{Br}(L)$.

Proof. Take $N = B$, the left regular representation of B, in Thm. 4.1. The second equality in 4.1 (vi) becomes

$$B^{\mathrm{opp}} \otimes_K \mathrm{Z}_B(A) \cong \mathrm{End}_A(B_s) \cong \mathrm{M}_{[B:A]}(A^{\mathrm{opp}})$$

because B_s is a free left A-module of rank $[B : A]$. Similarly the first equality in 4.1 (vi) reads

$$B^{\mathrm{opp}} \otimes_K A \cong \mathrm{M}_{[B:\mathrm{Z}_B(A)]}(\mathrm{Z}_B(A)^{\mathrm{opp}}) . \qquad \square$$

(4.3) Corollary *Let A be a finite dimensional central simple algebra over a field K, and let F be a subfield of A which contains K. Then F is a maximal subfield of A if and only if* $[F : K]^2 = [A : K]$.

Proof. Immediate from Thm. 4.1 (iii).

(4.4) Proposition *Let D be a finite dimensional central division algebra over a field K. Then D admits a maximal subfield L with* $[L : K]^2 = \dim_K(D)$ *such that L is separable over K. In particular D has a separable splitting field.*

Proof. Induction on $\dim_K(D)$, use Proposition 2.7 and Theorem 4.1 (iii). □

Remark It is not true that every finite dimensional central simple algebra A over K has subfield L with $[L : K]^2 = \dim_K(A)$. The most obvious example is when K is algebraically closed. Another similar example is when $K \supset \mathbb{F}_p$ is separably closed and $\dim_K(A)$ is relatively prime to p.

(4.5) Proposition *Let A be a finite dimensional central simple algebra over K. Let F be an extension field of K such that* $[F : K] = n := [A : K]^{1/2}$. *Then there exists a K-linear ring homomorphism* $F \hookrightarrow A$ *if and only if* $A \otimes_K F \cong \mathrm{M}_n(F)$.

Proof. The "only if" part is contained in Lemma 2.6. It remains to show the "if" part. Suppse that $A \otimes_K F \cong \mathrm{M}_n(F)$. Choose a K-linear embedding $\alpha : F \hookrightarrow \mathrm{M}_n(K)$. The central simple algebra $B := A \otimes_K \mathrm{M}_n(K)$ over K contains $C_1 := A \otimes_K \alpha(F)$ as a subalgebra, whose centralizer in B is $K \otimes_K \alpha(F)$. Since $C_1 \cong \mathrm{M}_n(F)$ by assumption, C_1 contains a subalgebra C_2 which is isomorphic to $\mathrm{M}_n(K)$. By Noether-Skolem, $\mathrm{Z}_B(C_2)$ is isomorphic to A over K. So we get $F \cong \mathrm{Z}_B(C_1) \subset \mathrm{Z}_B(C_2) \cong A$. □

(4.6) Corollary *Let Δ be a central division algebra over K, and let F be a finite extension field of K. Let $n = \dim_K(\Delta)^{1/2}$. The field F is a splitting field of Δ if and only if $n \mid [F : K]$ and F is a maximal subfield of $\mathrm{M}_r(\Delta)$, where $r = [F : K]/n$.*

Proof. By 4.5, it suffices to show that if F is a splitting field of Δ, then $n \mid [F : K]$. But then we have an action of Δ on $F^{\oplus n}$, and $n^2 = \dim_K(D) \mid \dim_K(F^{\oplus}) = n[F : K]$. Therefore $n \mid [F : K]$. □

Remark Here is an equivalent form of 4.6, and a direct proof.

> *Let A be a central simple algebra over a field K, and let L be a splitting field of A. Then there exists a central simple algebra A_1 in the same Brauer class of A which has a maximal subfield L_1 isomorphic to L over K.*

Proof. Let $\dim_K(A) = n^2$, $d = [L : K]$. By assumption we have

$$A^{\mathrm{opp}} \otimes_K L \xrightarrow[\sim]{\alpha} \mathrm{M}_n(L) \xhookrightarrow{j} \mathrm{M}_{nd}(K) =: B\,.$$

According to 4.1 (iv), $A_1 := \mathrm{Z}_C((j \circ \alpha)(A^{\mathrm{opp}} \otimes 1))$ is a central simple algebra over K, in the same Brauer class as A. Theorem 4.1 (iii) tells us that $\dim_K(A_1) = d^2 = [L : K]^2$. Clearly $L_1 := (j \circ \alpha)(1 \otimes L) \subset A_1$, so L_1 is a maxmial subfield of A_1. □