

Group (co)homologies. Summary

G : a group $\text{Mod}_G =$ the category of left $\mathbb{Z}[G]$ modules

$$1. M \in \text{Mod}_G \rightsquigarrow \begin{cases} H_i(G, M) = \text{Tor}_i^{\mathbb{Z}[G]}(M, \mathbb{Z}) & \leftarrow \text{trivial } G\text{-module } \forall i \geq 0 \\ H^i(G, M) = \text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, M) & \leftarrow \text{turned into a right } G\text{-module via } G \xrightarrow{\sim} G^{\text{opp}} \xrightarrow{\sigma \mapsto \sigma^{-1}} \end{cases}$$

$$0 \rightarrow I_G \rightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0 \quad \varepsilon: \sum_{\sigma \in G} n_\sigma [\sigma] \mapsto \sum_{\sigma \in G} n_\sigma$$

augmentation ideal

Explicit formula, from the bar resolution

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0(G) \xleftarrow{\partial_1} C_1(G) \xleftarrow{\partial_2} C_2(G) \leftarrow \dots \leftarrow C_{n-1}(G) \xleftarrow{\partial_n} C_n(G) \leftarrow \dots$$

$\mathbb{Z}[G]$ -module structure from the first factor

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{n-1}] \\ & & \downarrow \cup \\ & & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^n] \\ & & \downarrow \cup \\ & & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{n+1}] \end{array}$$

$$\begin{array}{ccc} \sigma_0 \cdot \sigma_1 [\sigma_2, \dots, \sigma_n] & \xleftarrow{\partial_n} & \sigma_0 \cdot [\sigma_1, \dots, \sigma_n] \\ + \sum_{i=1}^{n-1} (-1)^i \sigma_0 [\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_n] & & \\ + (-1)^n \sigma_0 [\sigma_1, \sigma_2, \dots, \sigma_{n-1}] & & \end{array}$$

$$\rightsquigarrow \begin{cases} H_i(G, M) = H_i(\dots \rightarrow C_n(G) \otimes_G M \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1(G) \otimes_G M \xrightarrow{\partial_1} C_0(G) \otimes_G M \rightarrow 0) \quad \forall i \geq 0 \\ C_n(G) \otimes_G M = C_n(G) \otimes_{\mathbb{Z}} M / \langle \sigma x \otimes \sigma m - x \otimes m \mid \sigma \in G, x \in C_n(G), m \in M \rangle \\ H^i(G, M) = H^i(0 \rightarrow \text{Hom}_{\mathbb{Z}[G]}(C_0(G), M) \xrightarrow{d^1} \dots \rightarrow \text{Hom}_{\mathbb{Z}[G]}(C_n(G), M) \xrightarrow{d^n} \dots) \\ \text{Maps}^n(G^n, M) \end{cases}$$

$$H_0(G, M) = M_G = M / I_G \cdot M, \quad H_1(G, \mathbb{Z}) = G^{ab} = G / (G, G)$$

\uparrow G -coinvariants

$$H_1(G, M) = G^{ab} \otimes_{\mathbb{Z}} M \quad \text{if } G \text{ operates trivially on } M$$

\downarrow G -invariants

$$H^0(G, M) = M^G, \quad H^1(G, M) = \text{Hom}_{\text{grp}}(G, M) \quad \text{if } G \text{ operates trivially on } M$$

In particular, $H^1(G, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{grp}}(G^{ab}, \mathbb{Q}/\mathbb{Z}) =$ Pontryagin dual of G^{ab}

$H^2(G, M) =$ equivalence classes of group extensions $1 \rightarrow M \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ such that $(M + \text{the conjugation action}) =$ the given G -module structure on M

(Recall that

$$0 \rightarrow I_G \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0$$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

$$\Rightarrow \begin{array}{l} H_i(G, \mathbb{Z}[G]) = 0 \quad \forall i \geq 1 \rightsquigarrow H_1(G, \mathbb{Z}) \cong H_0(G, I_G) = I_G / I_G^2 \\ \text{Assume } |G| = q < \infty. \text{ Then } H_i(G, \mathbb{Q}) = 0 \quad \forall i \geq 1, H^i(G, \mathbb{Q}) = 0 \quad \forall i \geq 1, \\ H^i(G, \mathbb{Z}[G]) = 0 \quad \forall i \geq 1, \\ H_1(G, \mathbb{Q}/\mathbb{Z}) = 0, H_1(G, \mathbb{Q}) = 0, H^i(G, \mathbb{Z}[G]) = 0 \\ H^1(G, \mathbb{Z}) = 0, H^2(G, I_G) = 0, \\ H^1(G, I_G) \cong \mathbb{Z}/q\mathbb{Z} \end{array}$$

2. Tate cohomology groups: Assume $|G| = g < \infty$

Let $C_{-r-1}(G) \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Z}}(C_r(G), \mathbb{Z}) \quad \forall r \geq 0$, with G -action by $(\sigma, \lambda)(x) = \lambda(\sigma^{-1}x)$
 $\leftarrow \in C_{-r-1}(G) \quad \forall x \in C_r(G)$

"complete resolution"

$$\dots \xleftarrow{\partial_3} C_{-3}(G) \xleftarrow{\partial_2} C_{-2}(G) \xleftarrow{\partial_1} C_{-1}(G) \xleftarrow{\partial_0} C_0(G) \xleftarrow{\partial_{-1}} C_1(G) \xleftarrow{\partial_{-2}} C_2(G) \xleftarrow{\dots} \dots$$

$$\hat{H}^i(G, M) \quad \forall i \in \mathbb{Z}$$

$$\hat{H}^i \left(\dots \xrightarrow{\text{deg}=-3} C_2(G) \otimes_{\mathbb{Z}} M \xrightarrow{\text{deg}=-2} C_1(G) \otimes_{\mathbb{Z}} M \xrightarrow{\text{deg}=-1} C_0(G) \otimes_{\mathbb{Z}} M \xrightarrow{d^0} \text{Hom}_{\mathbb{Z}}(C_0(G), M) \xrightarrow{d^1} \text{Hom}_{\mathbb{Z}}(C_1(G), M) \xrightarrow{d^2} \dots \right)$$

$\downarrow \quad \quad \quad \downarrow$
 $\mathbb{Z} \otimes_{\mathbb{Z}} M \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$
 $\parallel \quad \quad \quad \parallel$
 $M_G \xrightarrow{N_G} M^G$
 $N_G = \sum_{\sigma \in G} \sigma$ (called either norm or trace)

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & \text{if } i \geq 1 \\ H_{i-1}(G, M) & \text{if } i \leq -2 \end{cases}$$

$$\hat{H}^{-1}(G, M) = \text{Ker} (M_G \xrightarrow{N_G} M^G) = M[N_G] / \mathbb{Z} \cdot M$$

$$\hat{H}^0(G, M) = \text{Coker} (M_G \xrightarrow{N_G} M^G) = M^G / N_G \cdot M$$

Properties: If M is a projective $\mathbb{Z}[G]$ -module, then $\hat{H}^i(G, M) = 0 \quad \forall i \in \mathbb{Z}$

Special case: $G \cong \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle, n \geq 2$. Let $N = 1 + \sigma + \dots + \sigma^{n-1}$

\Rightarrow a periodic complete resolution

$$\dots \xleftarrow{N} \mathbb{Z}[G] \xleftarrow{[\sigma]-1} \mathbb{Z}[G] \xleftarrow{N} \mathbb{Z}[G] \xleftarrow{[\sigma]-1} \mathbb{Z}[G] \xleftarrow{N} \mathbb{Z}[G] \xleftarrow{[\sigma]-1} \mathbb{Z}[G] \xleftarrow{N} \dots$$

$\uparrow \quad \quad \quad \downarrow$
 $\mathbb{Z} = \mathbb{Z}$

$$\Rightarrow \hat{H}^i(G, M) \cong \hat{H}^{i+2}(G, M) \quad \forall i \in \mathbb{Z}$$

$$\begin{cases} \hat{H}^{\text{even}}(G, M) \cong M^G / N_G \cdot M \\ \hat{H}^{\text{odd}}(G, M) \cong M[N_G] / ([\sigma]-1) \cdot M \end{cases}$$

Definition (Herbrand quotient) $G \cong \mathbb{Z}/n\mathbb{Z}$, $G \cong \mathbb{Z}/n\mathbb{Z}$

$$h_{0,1}(M) = \frac{\# \hat{H}^0(G, M)}{\# \hat{H}^1(G, M)} \quad \text{if both } \hat{H}^0(G, M) \text{ and } \hat{H}^1(G, M) \text{ are finite}$$

Proposition $G \cong \mathbb{Z}/n\mathbb{Z}$,

(a) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ exact in Mod_G ,

then $h_{0,1}(M) = h_{0,1}(M') \cdot h_{0,1}(M'')$, and if 2 of the 3 terms are defined, so is the third.

(b) If $\# M < \infty$, then $h_{0,1}(M) = 1$

$$\left(\begin{array}{ccccccc} 0 & \rightarrow & M^G & \rightarrow & M & \xrightarrow{\sigma-1} & M & \rightarrow & M_G & \rightarrow & 0 \\ 0 & \rightarrow & \hat{H}^{-1}(G, M) & \rightarrow & M_G & \xrightarrow{N_G} & M^G & \rightarrow & \hat{H}^0(G, M) & \rightarrow & 0 \end{array} \right)$$

3. Change of groups

3.1 $\lambda: H \rightarrow G$ group homomorphism, M : left G -module $\Rightarrow \text{Res}_\lambda M$: left H -module

h induces a map of chain complexes $\lambda_*: C_*(H) \rightarrow C_*(G)$ $x_0 \otimes [x_1, \dots, x_i] \mapsto \lambda(x_0) \otimes [\lambda(x_1), \dots, \lambda(x_i)]$
 $\forall x_0, x_1, \dots, x_i \in H$

(a) $C_*(H) \otimes_H M \xrightarrow{\lambda_* \otimes \text{id}_M} C_*(G) \otimes_H M \rightarrow C_*(G) \otimes_G M$ induces

$H_i(\lambda_*) : H_i(H, M) \xrightarrow{\text{Res}_\lambda} H_i(G, M)$

(b) $\text{Hom}_G(C_*(G), M) \rightarrow \text{Hom}_H(C_*(G), M) \xrightarrow{\text{Hom}(\lambda_*, M)} \text{Hom}_H(C_*(H), M)$ induces

$H^i(\lambda^*) : H^i(G, M) \rightarrow H^i(H, M)$

Both are morphisms of δ -functors on $\text{Mod}_G =$ the category of all left G -modules

$\rightarrow H_0(\lambda_*)$ is determined by $H_0(\lambda_*) : M_H \rightarrow M_G$

$H^1(\lambda^*)$ is determined by $H^1(\lambda^*) : M^G \rightarrow M^H$

3.2 When $H \leq G$, $N \in \text{ob}(\text{Mod}_H)$, have

$C_*(G) \otimes_G (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N) \cong C_*(G) \otimes_{\mathbb{Z}[H]} N$
 $\text{ind}_H^G N$ $\text{Res}_H^G(C_*(G)) \leftarrow$ a free resolution of \mathbb{Z} in Mod_H

$\text{Hom}^G(C_*(G), \text{Ind}_H^G N) \cong \text{Hom}^H(\text{Res}_H^G C_*(G), N)$
 $\{f: G \rightarrow N \mid f(hx) = h \cdot f(x) \forall h \in H, \forall x \in G\}$

$\Rightarrow \begin{cases} H_i(G, \text{ind}_H^G N) \cong H_i(H, N) \\ H^i(G, \text{Ind}_H^G N) \cong H^i(H, N) \end{cases} \forall i \geq 0$ "Shapiro's Lemma"

$M \in \text{Mod}_G \Rightarrow \begin{cases} H_*(H \hookrightarrow G)_* M = H_* \left[C_*(G) \otimes_{\mathbb{Z}[G]} \left(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \right) \rightarrow M \right] \\ H^*(H \hookrightarrow G)^* M = H^* \left[\text{Hom}_G(C_*(G), M) \rightarrow \text{Ind}_H^G \text{Res}_H^G M \right] \\ m \mapsto (x \mapsto x \cdot m) \end{cases}$

3.3. If $H \leq G$ and $[G:H] = a < \infty$, have transfer/corestriction maps

$M \in \text{Mod}_G \quad H_i(G, M) \xrightarrow{\text{Ver}} H_i(H, M) \quad \text{and similarly} \quad \hat{H}^i(G, M) \xleftarrow{\text{Ver}} \hat{H}^i(H, M)$
 $H^i(G, M) \xleftarrow{\text{Ver}} H^i(H, M) \quad \text{if } |G| < \infty$

Defining properties of transfer

For H^* : $H^0(H, M) = M^H \xrightarrow{N_{G/H}} M^G = H^0(G, M)$
 $m \mapsto \sum_{x \in G/H} x \cdot m$

For H_* : $H_0(G, M) = M_G \xrightarrow{N_{HG}} M_H = H_0(H, M)$ \leftarrow Well-defined in $M_{\mathbb{Z}}$
 $m \text{ mod } I_G M \mapsto \sum_{x \in H \backslash G} x \cdot m \text{ mod } I_H M$

(x_i) system of representatives of $H \backslash G$
 $\sigma \in G$. Write $x_i \cdot \sigma = h_{\pi(i)} \cdot x_{\pi(i)}$
 \uparrow permutation of $H \backslash G$, depending on σ

In $\mathbb{Z}[G]$, have

$$\sum_i x_i \cdot (\sigma^{-1}) = \sum_i h_{\pi(i)} x_{\pi(i)} - \sum_i x_i = \sum_i (h_i - 1) \cdot \sigma \in I_H \cdot \mathbb{Z}[G]$$

Have

$$H^i(G, M) \xrightarrow{\text{Res}} H^i(H, M) \xrightarrow{\text{Ver}} H^i(G, M) \quad \text{check } H^0$$

$\underbrace{\hspace{10em}}_{[G:H]} \uparrow$

$$H_i(G, M) \xrightarrow{\text{Ver}} H_i(H, M) \xrightarrow{(H \hookrightarrow G)_*} H_i(G, M) \quad \text{check } H_0$$

$\underbrace{\hspace{10em}}_{[G:H]} \uparrow$

Explicit formula for a quasi-isom of complexes in Mod_H

$$\text{Res}_H^G C(G) \xrightarrow{\text{q. isom}} C(H) \quad \text{in Assignment 13}$$

3.3. Restriction - inflation sequence (for a normal subgroup)

$$N \triangleleft G$$

$$M \in \text{Mod}_G$$

$$0 \rightarrow H^i(G/N, M^N) \xrightarrow{\text{Inf}} H^i(G, M) \xrightarrow{\text{Res}} H^i(N, M)$$

$$0 \leftarrow H_i(G/N, M_N) \leftarrow H_i(G, M) \leftarrow H_i(N, M)$$

are exact
when $i=1$,
(direct computation)

and for $i \neq 1$ if $H^i(G, M) = 0$
 $H_j(G, M) = 0$

for $1 \leq j \leq q-1$

(either by dimension shifting
or use Hochschild-Serre s. seq.)
 $E_2^{i,j} = H^i(G/N, H^j(N, M))$
 $\Rightarrow H^{i+j}(G, M)$

4. Cup product

characterization by functorial properties $G = \text{finite group}$
 $M, N \in \text{Mod}_G$

$$\hat{H}^i(G, M) \times \hat{H}^j(G, N) \longrightarrow \hat{H}^{i+j}(G, M \otimes_{\mathbb{Z}} N)$$

(alternative notation: $a \cup b$)

(i) functorial in M and N

(ii) When $i=j=0$, it is induced by the natural map

$$M^G \otimes_{\mathbb{Z}} N^G \longrightarrow (M \otimes_{\mathbb{Z}} N)^G$$

(iii) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact and $0 \rightarrow M' \otimes_{\mathbb{Z}} N \rightarrow M \otimes_{\mathbb{Z}} N \rightarrow M'' \otimes_{\mathbb{Z}} N \rightarrow 0$ is also exact, then $\forall a' \in \hat{H}^i(G, M')$, $\forall b \in \hat{H}^j(G, N)$, we have

$$\underbrace{(\delta a')}_{\hat{H}^{i+1}(G, M')} \cdot b = \delta \underbrace{(a' \cdot b)}_{\hat{H}^{i+j}(G, M' \otimes_{\mathbb{Z}} N)} \in \hat{H}^{i+j+1}(G, M \otimes_{\mathbb{Z}} N)$$

(iv) If $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ is exact and

$0 \rightarrow M \otimes_{\mathbb{Z}} N' \rightarrow M \otimes_{\mathbb{Z}} N \rightarrow M \otimes_{\mathbb{Z}} N'' \rightarrow 0$ is also exact, $a \in \hat{H}^i(G, M)$
 $b \in \hat{H}^j(G, N')$

$$\text{then } (-1)^j \cdot a \cdot \underbrace{\delta b'}_{\hat{H}^{j+1}(G, N')} = \delta \underbrace{(a \cdot b'')}_{\hat{H}^{i+j}(G, M \otimes_{\mathbb{Z}} N'')} \in \hat{H}^{i+j+1}(G, M \otimes_{\mathbb{Z}} N')$$

Remark Have explicit chain map $C_*(G) \xrightarrow{\Phi} C_*(G) \otimes_{\mathbb{Z}} C_*(G)$
 (a co-pairing)
 which induces the cup product

[Note By defⁿ, Φ is a collection of maps $C_{i+j}(G) \xrightarrow{\phi_{i,j}} C_i(G) \otimes_{\mathbb{Z}} C_j(G)$
 $i, j \in \mathbb{Z}$
 (It is not a map from the \mathbb{Z} -module $C_*(G)$ to the module $C_*(G) \otimes_{\mathbb{Z}} C_*(G)$)

5. Cohomologically trivial modules for finite groups — Theorems of Tate and Nakayama

Prop. 5.1. (p -groups) Let G be a finite p -group. Let M be a G -module

(a) Suppose that $p \cdot M = 0$. If $\hat{H}^q(G, M) = 0$ for some $q_0 \in \mathbb{Z}$. Then M is a free $\mathbb{F}_p[G]$ -module, and $\hat{H}^q(K, M) = 0$ for all $q \in \mathbb{Z}$ and every subgroup $K \leq G$.

(b) Suppose that M is torsion-free, and $\hat{H}^q(G, M) = 0 = \hat{H}^{q+1}(G, M) = 0$ for some $q_0 \in \mathbb{Z}$. Then

(b1) $\hat{H}^q(K, M) = 0 \quad \forall q \in \mathbb{Z}, \forall K \leq M$

(b2) M/pM is a free $\mathbb{F}_p[G]$ -module

(b3) Assume that M is a free \mathbb{Z} -module \downarrow torsion-free G -module N , we have $\hat{H}^q(K, \text{Hom}_{\mathbb{Z}}(M, N)) = 0 \quad \forall q \in \mathbb{Z}, \forall K \leq G$

Will see in thm 5.2 that: (b4) \exists a projective resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ of length 1 of M in Mod_G

Pf. (a) dimension shift $\rightsquigarrow \exists$ a G -module N s.t. $p \cdot N = 0$ and $\hat{H}^{n-q_0-2}(G, N) \cong \hat{H}^n(G, M) = 0 \quad \forall n \in \mathbb{Z}$.

Assumption: $\hat{H}^{-2}(G, N) = H_1(G, N) = 0$

Let $L \xrightarrow{\alpha} N$ be a G -linear surjection s.t. $\begin{cases} L = \mathbb{F}_p[G] \otimes_{\mathbb{F}_p} L \text{ (an } \mathbb{F}_p\text{-v.sp.)} \\ L/I_G L \xrightarrow{\alpha} N/I_G N \end{cases}$

$0 \rightarrow Q \rightarrow L \xrightarrow{\alpha} N \rightarrow 0$ + long exact sequence $\Rightarrow H_0(G, Q) = Q/I_G Q = 0$, i.e. $I_G \cdot Q = Q$

Easy Fact/Exer: $\exists m_0 \in \mathbb{N}$ s.t. $(\mathbb{F}_p I_G)^{m_0} = 0$. (E.g. induction on $|G|$)

$\rightsquigarrow Q = I_G Q = I_G^2 Q = \dots = I_G^{m_0} Q = 0$, i.e. $L \xrightarrow{\sim} N \Rightarrow \hat{H}^q(G, M) = 0 \quad \forall q \in \mathbb{Z}$
 $\Rightarrow M$ is a free $\mathbb{F}_p[G]$ -module.

(b) $0 \rightarrow M \xrightarrow{p} M \rightarrow M/pM \rightarrow 0 \rightsquigarrow$ Assumption $\Rightarrow \hat{H}^{q_0}(G, M/pM) = 0$, hence M/pM is a free $\mathbb{F}_p[G]$ -module.

Assume now that M is a free \mathbb{Z} -module. Consider $0 \rightarrow N \xrightarrow{p} N \rightarrow N/pN \rightarrow 0$ exact

$\Rightarrow 0 \rightarrow \text{Hom}_{\mathbb{Z}}(M, N) \xrightarrow{p} \text{Hom}_{\mathbb{Z}}(M, N) \rightarrow \text{Hom}_{\mathbb{Z}}(M, N/pN) \rightarrow 0$

Key observation: \leftarrow is a free $\mathbb{F}_p[G]$ -module $\rightarrow \text{Hom}_{\mathbb{F}_p}(M/pM, N/pN)$

$M/pM = \bigoplus_{i \in I} \mathbb{F}_p[G] \cdot e_i \Rightarrow \bigoplus_{i \in I} \mathbb{F}_p[G] \cdot \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p \cdot e_i, N/pN)$

$\Rightarrow \hat{H}^q(K, \text{Hom}_{\mathbb{Z}}(M, N)) \xrightarrow{p} \hat{H}^q(K, \text{Hom}_{\mathbb{Z}}(M, N)) \quad \forall q \in \mathbb{Z}, \forall K \leq G$

killed by $|G| \in \mathbb{F}_p^{\times}$

QED.

(Nakayama 1957)

Thm 5.2 G : a finite group, M : a G -module. ^{a Sylow p -subgroup of G}
 Assume that \forall prime $p \mid \#G, \exists q(p) \in \mathbb{Z}$ s.t. $\hat{H}^{q(p)}(G_p, M) = 0 = \hat{H}^{q(p)+1}(G, M)$

(a) $\hat{H}^q(K, M) = 0 \quad \forall q \in \mathbb{Z}, \forall K \leq G$

(b) \exists a G -linear projective resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ of length = 1

(c) If M is a free \mathbb{Z} -module, then M is a projective $\mathbb{Z}[G]$ -module

($\Leftrightarrow M$ is a projective \mathbb{Z} -module)

Pf. of (c): Pick a short exact sequence $0 \rightarrow Q \rightarrow L \rightarrow M \rightarrow 0$ in Mod_G .

Consider the exact sequence

$$0 \rightarrow \text{Hom}_{\mathbb{Z}}(M, Q) \rightarrow \text{Hom}_{\mathbb{Z}}(M, L) \rightarrow \text{Hom}_{\mathbb{Z}}(M, M) \rightarrow 0$$

Note: Not every torsion free \mathbb{Z} -module is projective: every projective \mathbb{Z} -module is free; Q is a torsion free and flat \mathbb{Z} -module but not a free \mathbb{Z} -module.

$\circ M$ is a free \mathbb{Z} -module

$$\hat{H}^1(G_p, \text{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \forall p \mid \#G$$

$$\Rightarrow \hat{H}^1(G, \text{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \circ \text{Ker}(\hat{H}^1(G, \text{Hom}_{\mathbb{Z}}(M, Q)) \xrightarrow{\text{restriction}} \hat{H}^1(G_p, \text{Hom}_{\mathbb{Z}}(M, Q)))$$

$$\Rightarrow M \text{ is a direct summand of } L, \text{ hence projective} = \hat{H}^1(G, \text{Hom}_{\mathbb{Z}}(M, Q)) [G:G_p] = \{ h \in \hat{H}^1(G, \text{Hom}_{\mathbb{Z}}(M, Q)) \mid [G:G_p]h = 0 \}$$

(a)+(b): Pick a short exact sequence $0 \rightarrow R \rightarrow L \rightarrow M \rightarrow 0$ in Mod_G .

R is a free \mathbb{Z} -module, and $\hat{H}^{q(p)+1}(G_p, R) = \hat{H}^{q(p)+2}(G_p, R) \quad \forall p \mid \#G$

Every submodule of a free module over a PID is free. $\Rightarrow R$ is a projective $\mathbb{Z}[G]$ -module by (c).

QED.

Cor. 5.3 G : a finite group, B, C : G -modules. $f: B \rightarrow C$ G -linear (mapping cone construction) Suppose that \forall prime $p \mid \#(G), \exists q(p) \in \mathbb{Z}$ s.t.

$f_q^*: \hat{H}^q(G_p, B) \rightarrow \hat{H}^q(G_p, C)$ is surjective for $q = q(p)$,

bijection for $q = q(p)+1$ and injective for $q = q(p)+2$. Then $\forall q \in \mathbb{Z}$ and $\forall K \leq G$,

$$f_q^*: \hat{H}^q(K, B) \xrightarrow{\sim} \hat{H}^q(K, C)$$

Pf: $0 \rightarrow B \rightarrow C \oplus \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], B) \rightarrow D \rightarrow 0$ short exact

$$b \mapsto (f(b), \varphi = (x \mapsto x \cdot b)_{x \in G})$$

Assumption $\Rightarrow \hat{H}^{q(p)}(G_p, D) = 0 = \hat{H}^{q(p)}(G_p, D) \quad \forall p \mid \#D$

$$\xrightarrow{\text{Thm 5.2}} \hat{H}^q(K, D) = 0 \quad \forall q \in \mathbb{Z} \Rightarrow \hat{H}^q(K, B) \xrightarrow{f_q^*} \hat{H}^q(K, C)$$

Prop. 5.4 $A, B, C \in \text{Mod}_G$, $\varphi: A \otimes_{\mathbb{Z}} B \rightarrow C$ G -linear

Given $\alpha \in \hat{H}^q(G, A)$. Assume \forall prime $p \mid \#G$, $\exists n(p) \in \mathbb{Z}$ s.t.
 the map $\hat{H}^{n(p)}(G_p, B) \xrightarrow{\beta} \hat{H}^{n(p)+q}(G_p, C)$ is surjective for $n=n(p)$,

bijjective for $n=n(p)+1$ and injective for $n=n(p)+2$. Then for every $K \leq G$ and every $n \in \mathbb{Z}$, the map

$$\begin{array}{ccc} \hat{H}^n(K, B) & \xrightarrow{\beta} & \hat{H}^{n+q}(K, C) \\ \downarrow \beta & \longmapsto & \downarrow \beta \\ \beta & \longmapsto & \text{Res}_K^G(\alpha) \cup \beta \end{array}$$

is an isomorphism.

Pf. The case $q=0$ follows from Cor. 5.3.

shift dimension with diagrams

$$\begin{array}{l} 0 \rightarrow A' \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \rightarrow A \rightarrow 0 \quad + \quad 0 \rightarrow C' \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} C \rightarrow C \rightarrow 0 \quad + \quad A' \otimes_{\mathbb{Z}} B \rightarrow C' \\ \text{and} \\ 0 \rightarrow A \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \rightarrow A' \rightarrow 0, \quad 0 \rightarrow C \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], C) \rightarrow C' \rightarrow 0 \quad + \quad A' \otimes_{\mathbb{Z}} B \rightarrow C' \end{array}$$

Theorem 5.5 (Tate 1952) Let M be a G -module, and let $\alpha \in H^2(G, M)$

Assume (i) $H^1(G_p, M) = 0$

(ii) $H^2(G_p, M) = \mathbb{Z} \cdot \text{Res}_{G \geq G_p}(\alpha) \cong \mathbb{Z} / \#G_p \cdot \mathbb{Z} \quad \forall \text{ prime } p \mid \#G$.

Then $\hat{H}^n(K, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^{n+2}(K, M) \quad \forall n, \forall K \leq G$.

Immediate Corollary of Prop. 5.4, with $A=M, B=\mathbb{Z}, n(p)=-1 \quad \forall p$

Note that $H^1(G_p, \mathbb{Z})=0$, and the injectivity of $H^1(G_p, \mathbb{Z}) \rightarrow H^3(G_p, M)$ holds trivially.

The case important for application is the isomorphism

$$K^{ab} = \hat{H}^{-2}(K, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^0(K, M) = M^K / N_K \cdot M$$