Math 602 Final Exam, December 2010

Instructor: Dr. Chai

December 8, 2010, 10:30 – 12:00 noon

This examination consists of five questions (with parts) and one extra-credit question. Please *show all your work* and *fully justify* your answers.

- Your name, printed:
- Your signature:

	Score
1 (20 pts)	
2 (25 pts)	
3 (25 pts)	
4 (15 + 10 extra)	
5 (15 + 10 extra)	
6 (20 extra pts)	
Total	

1. (20 pts) Let n be a positive integer. Let R be a commutative ring. Suppose that $\underline{a} = (a_1, \ldots, a_n)$ is an element of R^n such that $\underline{a} \notin \mathfrak{m}R^n$ for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$. Prove that the R-module $R \cdot \underline{a}$ is a direct summand of the free R-module R^n.

2. Recall that for any commutative ring R, $R[[x]]$ stands for the ring of formal power series in one variable x with coefficients in R, consisting of all formal expressions

$$\sum_{n\in\mathbb{N}} a_n\, x^n\,, \qquad a_n \in R \;\; \forall n \in \mathbb{N}\,;$$

addition and multiplication in the ring $R[[x]]$ are defined by the usual formulas.

(a) (10 pts) Show that the natural inclusion $\mathbb{Z}[[x]] \hookrightarrow \mathbb{Q}[[x]]$ induces an injective ring homomorphism $h\colon \mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}[[x]]$.

(b) (15 pts) Determine whether the ring extension $h\colon \mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \mathbb{Q}[[x]]$ is integral.

3. (25 pts) Let p be a prime number. Find an injective envelope of the $(\mathbb{Z}/p^5\mathbb{Z})$-module $\mathbb{Z}/p\mathbb{Z}$.

4. For any module M over a commutative ring R, the *support* $\mathrm{supp}_R(M)$ is defined to be the subset of $\mathrm{Spec}(R)$ consisting of all prime ideals $\mathfrak{p}$ in R such that the localization $M_{\mathfrak{p}} \neq 0$.

(a) (15 pts) Suppose that M is a finitely generated R-module. Prove that $\mathrm{supp}_R(M)$ is a closed subset of $\mathrm{Spec}(R)$.

(b) (10 extra pts) Is the statement (a) true without the assumption that M is finitely generated? Either give a proof or a counter-example.

5. Let M and N be finitely generated modules over a commutative ring R.

   (a) (15 pts) Prove that $\operatorname{supp}_R(M \otimes_R N) = \operatorname{supp}_R(M) \cap \operatorname{supp}_R(N)$.

   (b) (10 extra points) Does the statement (a) hold if we assume only that M is finitely generated over R? Either give a proof or a counter-example.

6. (extra credit) Let R be the ring $\mathbb{C}[x, y]/(y^2 - x^3 + x)$.

(a) (10 extra points) Show that for every maximal ideal $\mathfrak{m}$ of R, the maximal ideal of the local ring $R_{\mathfrak{m}}$ is principal.

(b) (10 extra points) Let $I = x\mathbb{C}[x, y] + y\mathbb{C}[x, y]/(y^2 - x^3 + x)$ be the ideal of R generated by the image in R of x and y. Determine whether I is a principal ideal of R.