Part I. For each of the following, either give an example, or show that none exists.
1. (10 pts) A non-cyclic abelian group with 42 elements.
2. (10 pts) A group of order 16 which has no subgroup of order 4.
3. (10 pts) A solvable group which is not nilpotent.
4. (10 pts) A non-split extension of groups, i.e. a surjective homomorphism of groups \( \pi : G \rightarrow Q \) such that there is no homomorphism \( \alpha : Q \rightarrow G \) with \( \pi \circ \alpha = \text{Id}_Q \).
5. (10 pts) A surjective \( \mathbb{R} \)-linear homomorphism \( \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \rightarrow \mathbb{R}^3 \).
6. (10 pts) A ring \( R \), a left \( R \)-module \( N \) and a short exact sequence of left \( R \)-modules
\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]
such that the induced sequence of abelian groups
\[
0 \rightarrow \text{Hom}_R(M'', N) \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M', N) \rightarrow 0
\]
is not exact.
7. (10 pts) A ring \( R \), a left \( R \)-module \( N \) and a short exact sequence of left \( R \)-modules
\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]
such that the induced sequence of abelian groups
\[
0 \rightarrow M' \otimes_R N \rightarrow M \otimes_R N \rightarrow M'' \otimes_R N \rightarrow 0
\]
is not exact.
8. (15 pts) A prime ideal \( I \subset \mathbb{Q}[\mathbb{Z}/4\mathbb{Z}] =: R \) such that \( \dim_{\mathbb{Q}}(R/I) \geq 2 \).
9. (15 pts) A maximal ideal in \( \mathbb{R}[x, y]/(x^2 + y^2 + 1) \).

Part II.

10. (20 pts) (a) Determine the cardinality of the groups \( \text{Aut}_{\text{grp}}((\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z})) \).
(b) Is the group \( \text{Aut}_{\text{grp}}((\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z})) \) cyclic? (Either prove that it is cyclic, or prove that it is not cyclic.)

11. (15 pts) Let \( M \) be the \( \mathbb{Z} \)-module \( (\mathbb{Z}/10\mathbb{Z}) \oplus (\mathbb{Z}/25\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \). Determine the \( \mathbb{Z} \)-modules \( \bigwedge_3^\mathbb{Z} M \).

11a. (15 extra credit pts) Find the structure of the \( \mathbb{Z} \)-module \( \bigwedge_3^\mathbb{Z} M \).

12. (20 extra credit pts) Is there a finitely generated non-trivial \( \mathbb{Z} \)-module \( M \) such that there exist a natural number \( N_0 \in \mathbb{N} \) such that \( S_i^\mathbb{Z} M = (0) \) for all \( i \geq N_0 \)? ( Either give an example, or show that none exists.)