Chapter 2

Rings and Modules

2.1 Introduction

Linear algebra—meaning vector space theory over a field—is the part of algebra used most often in analysis, in geometry and in various applied fields. The natural generalization to the case when the base object is a ring rather than a field is the notion of "module." The theory of modules both delineates in sharp relief the elementary and deeper structure of vector spaces (and their linear transformations) and provides the essential "linear springboard" to areas such as number theory, algebraic geometry and functional analysis. It turns out to be surprisingly deep because the collection of "all" modules over a fixed ring has a profound influence on the structure of that ring. For a commutative ring, it even specifies the ring! Just as in analysis, where the first thing to consider in analyzing the local behavior of a given smooth function is its linear approximation, so in geometric applications the first idea is to pass to an appropriate linear approximation and this is generally a module.

2.2 Polynomial Rings, Commutative and Noncommutative

Consider the categories RNG and CR, and pick some ring, A, from each. We also have the category, RNG^A, called the *category of rings over* A (or *category of* A*-algebras*), and similarly, CR^A, and we have the stripping functors $\mathrm{RNG}^A \rightsquigarrow \mathcal{S}\mathrm{ets}$ and $\mathrm{CR}^A \rightsquigarrow \mathcal{S}\mathrm{ets}$.

Is there an adjoint functor to each? We seek a functor, $P\colon \mathcal{S}\mathrm{ets} \rightsquigarrow \mathcal{C}$, where $\mathcal{C} = \mathrm{RNG}^A$ or CR^A, so that

$$\mathrm{Hom}_{\mathcal{C}}(P(S), B) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |B|)$$

for every $B \in \mathcal{C}$.

Case 1: CR^A.

Theorem 2.1 *There exists a left-adjoint functor to the stripping functor,* $\mathrm{CR}^A \rightsquigarrow \mathcal{S}\mathrm{ets}$.

Proof. Given a set, S, let $\widetilde{\mathbb{N}}$ denote the set of non-negative integers and write $\widetilde{\mathbb{N}}_S$ for

$$\widetilde{\mathbb{N}}_S = \{\xi\colon S \to \widetilde{\mathbb{N}} \mid \xi(s) = 0, \quad \text{except for finitely many } s \in S\}.$$

Note that $\mathbb{N}_S$ consists of the functions $S \longrightarrow \widetilde{\mathbb{N}}$ with compact support (where S and $\widetilde{\mathbb{N}}$ are given the discrete topology).

Remark: We may think of the elements, ξ, of $\widetilde{\mathbb{N}}_S$ as finite *multisets* of elements of S, *i.e.*, finite sets with multiple occurrences of elements: For any $s \in S$, the number $\xi(s)$ is the number of occurrences of s in ξ. If we think of each member, s, of S as an "indeterminate," for any $\xi \in \widetilde{\mathbb{N}}_S$, if $\xi(s_i) = n_i > 0$ for $i = 1, \ldots, t$, then ξ corresponds to the monomial $s_1^{n_1} \cdots s_t^{n_t}$.

We define a multiplication operation on $\widetilde{\mathbb{N}}_S$ as follows: For $\xi, \eta \in \widetilde{\mathbb{N}}_S$,

$$(\xi\eta)(s) = \xi(s) + \eta(s).$$

(This multiplication operation on $\widetilde{\mathbb{N}}_S$ is associative and has the identity element, ξ_0, with $\xi_0(s) = 0$ for all $s \in S$. Thus, $\widetilde{\mathbb{N}}_S$ is a *monoid*. Under the interpretation of elements of $\widetilde{\mathbb{N}}_S$ as multisets, multiplication corresponds to union and under the interpretation as monomials, it corresponds to the intuitive idea of multiplication of monomials. See below for precise ways of making these intuitions correct.)

Define $A[S]$ by

$$A[S] = \{f\colon \widetilde{\mathbb{N}}_S \to A \mid f(\xi) = 0, \quad \text{except for finitely many } \xi \in \widetilde{\mathbb{N}}_S\}.$$

Remark: We should think of each $f \in A[S]$ as a polynomial in the indeterminates, s ($s \in S$), with coefficients from A; each $f(\xi)$ is the coefficient of the monomial ξ. See below where X_s is defined.

In order to make $A[S]$ into a ring, we define addition and multiplication as follows:

$$\begin{aligned} (f+g)(\xi) &= f(\xi) + g(\xi) \\ (fg)(\xi) &= \sum_{\substack{\eta,\eta',\\ \eta\eta'=\xi}} f(\eta)g(\eta'). \end{aligned}$$

Multiplication in $A[S]$ is also called the *convolution product*. The function with constant value, $0 \in A$, is the zero element for addition and the function denoted 1, given by

$$1(\xi) = \begin{cases} 0 & \text{if } \xi \neq \xi_0 \\ 1 & \text{if } \xi = \xi_0, \end{cases}$$

is the identity element for multiplication. The reader should check that under our operations, $A[S]$ is a commutative ring with identity (DX). For example, we check that 1 is an identity for multiplication. We have

$$(f \cdot 1)(\xi) = \sum_{\eta\eta'=\xi} f(\eta)1(\eta') = \sum_{\eta\xi_0=\xi} f(\eta).$$

However, for all $s \in S$, we have $\eta\xi_0(s) = \eta(s) + \xi_0(s) = \eta(s)$, and so, $\eta = \xi$. Consequently, $(f \cdot 1)(\xi) = f(\xi)$, for all ξ.

We have an injection $A \longrightarrow A[S]$ *via* $\alpha \in A \mapsto \alpha \cdot 1$. Here, $\alpha \cdot 1$ is given by

$$\alpha \cdot 1(\xi) = \alpha(1(\xi)) = \begin{cases} 0 & \text{if } \xi \neq \xi_0 \\ \alpha & \text{if } \xi = \xi_0. \end{cases}$$

Therefore, $A[S] \in \mathrm{CR}^A$. It remains to check the "universal mapping property."

Say $\theta \in \mathrm{Hom}_{\mathrm{CR}^A}(A[S], B)$. Now, we can define two injections $S \hookrightarrow \widetilde{\mathbb{N}}_S$ and $S \hookrightarrow A[S]$ (a map of sets) as follows: Given any $s \in S$, define $\Delta_s \in \widetilde{\mathbb{N}}_S$ by

$$\Delta_s(t) = \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s, \end{cases}$$

and define $X_s \in A[S]$ by

$$X_s(\xi) = \begin{cases} 0 & \text{if } \xi \neq \Delta_s \\ 1 & \text{if } \xi = \Delta_s. \end{cases}$$

Then, if we set $\theta^\flat(s) = \theta(X_s)$, we get a set map $\theta^\flat \in \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |B|)$.

Conversely, let $\varphi \in \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |B|)$. Define $\widetilde{\varphi}\colon \widetilde{\mathbb{N}}_S \to B$ *via*

$$\widetilde{\varphi}(\xi) = \prod_{s\in S} \varphi(s)^{\xi(s)} \in B.$$

Now, set $\varphi^\sharp(f)$, for $f \in A[S]$, to be

$$\varphi^\sharp(f) = \sum_{\xi} f(\xi)\widetilde{\varphi}(\xi).$$

(Of course, since $B \in \mathrm{CR}^A$, we view $f(\xi)$ as an element of B *via* the corresponding morphism $A \longrightarrow B$.)

The reader should check (DX) that:

(a) $\varphi^\sharp$ is a homomorphism and

(b) The operations $\sharp$ and $\flat$ are mutual inverses. □

The definition of $A[S]$ has the advantage of being perfectly rigorous, but it is quite abstract. We can give a more intuitive description of $A[S]$. For this, for any $\xi \in \widetilde{\mathbb{N}}_S$, set

$$X^{(\xi)} = \prod_{s\in S} X_s^{\xi(s)}, \quad \text{in } A[S],$$

and call it a *monomial*. The reader should check (DX) that

$$X^{(\xi)}(\eta) = \delta_{\xi\eta}, \quad \text{for all } \xi, \eta \in \widetilde{\mathbb{N}}_S.$$

Hence, the map $\xi \mapsto X^{(\xi)}$ is a bijection of $\widetilde{\mathbb{N}}_S$ to the monomials (c.f. the remark on monomials made earlier). Moreover, we claim that every $f \in A[S]$ can be written as

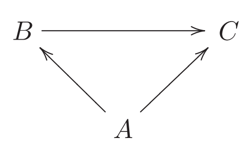
$$f = \sum_{\xi} f(\xi) X^{(\xi)}.$$

This is because

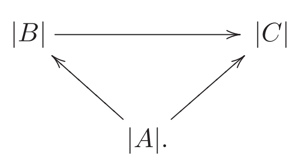
$$\left(\sum_{\xi} f(\xi) X^{(\xi)}\right)(\eta) = \sum_{\xi} f(\xi)\delta_{\xi\eta} = f(\eta).$$

The usual notation for $\xi(s)$ is ξ_s, and then, $X^{(\xi)} = \prod_{s\in S} X_s^{\xi_s}$, and our f's in $A[S]$ are just polynomials in the usual sense, as hinted at already. However, since S may be infinite, our formalism allows us to deal with polynomials in infinitely many indeterminates. Note that any polynomial involves just a finite number of the variables.

What happened to $|A|$ in all this? After all, in CR^A, we have rings, B, and maps $i_A\colon A \to B$. So, the commutative diagram



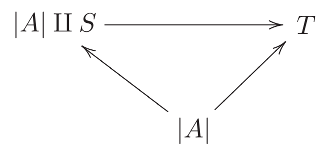
would give



Consider the category of $|A|$-sets, $\mathcal{S}\mathrm{ets}^{|A|}$. Given any set, S, make an $|A|$-set:

$$|A| \amalg S = |A| \cup S.$$

This is an $|A|$-set, since we have the canonical injection, $|A| \longrightarrow |A| \amalg S$. Let T be any $|A|$-set and look at $\mathrm{Hom}_{\mathcal{S}\mathrm{ets}^{|A|}}(|A| \amalg S, T)$, i.e., maps $|A| \amalg S \longrightarrow T$ such that the diagram



commutes. We know that

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ets}^{|A|}}(|A| \amalg S, T) \subseteq \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(|A|, T) \prod \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, T)$$

and the image is $\mathrm{Hom}_{\mathcal{S}\mathrm{ets}^{|A|}}(|A|, T) \prod \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, T)$. But $\mathrm{Hom}_{\mathcal{S}\mathrm{ets}^{|A|}}(|A|, T)$ consists of a single element, and so,

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ets}^{|A|}}(|A| \amalg S, T) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, T).$$

Thus, we have the functorial isomorphism

$$\mathrm{Hom}_{\mathrm{CR}^A}(A[S], B) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}^{|A|}}(|A| \amalg S, |B|).$$

Corollary 2.2 *A necessary and sufficient condition that $\mathbb{Z}[S] \cong \mathbb{Z}[T]$ (in* CR*) is that $\#(S) = \#(T)$.*

Proof. If $\#(S) = \#(T)$, then there exist mutually inverse bijections, $\varphi \colon S \to T$ and $\psi \colon T \to S$. Hence, by functoriality, $\mathbb{Z}[S]$ is isomorphic to $\mathbb{Z}[T]$ (*via* $\mathbb{Z}[S](\varphi)$ and $\mathbb{Z}[T](\psi)$). Now, take $B = \mathbb{Z}/2\mathbb{Z}$, and assume that $\mathbb{Z}[S] \cong \mathbb{Z}[T]$. Then, we know that

$$\mathrm{Hom}_{\mathrm{CR}}(\mathbb{Z}[S], B) \cong \mathrm{Hom}_{\mathrm{CR}}(\mathbb{Z}[T], B),$$

and since $\mathrm{Hom}_{\mathrm{CR}}(\mathbb{Z}[S], B) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, \{0,1\})$ and $\mathrm{Hom}_{\mathrm{CR}}(\mathbb{Z}[T], B) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(T, \{0,1\})$, we have

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, \{0,1\}) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(T, \{0,1\}).$$

This implies that $2^{\#(S)} = 2^{\#(T)}$, and thus, $\#(S) = \#(T)$. $\square$

Case 2: RNG^R, where R is a given ring (not necessarily commutative). For every set, S, and every R-algebra, $B \in \mathrm{RNG}^R$, let

$$\mathrm{Hom}^{(c)}_{\mathcal{S}\mathrm{ets}}(S, |B|) = \{\varphi \in \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |B|) \mid (\forall s \in S)(\forall \xi \in \mathrm{Im}\,(|R|))(\varphi(s)\xi = \xi\varphi(s))\}.$$

Theorem 2.3 *There exists a functor, $R\langle S\rangle$, from* $\mathcal{S}$ets *to* RNG^R*, so that*

$$\mathrm{Hom}_{\mathrm{RNG}^R}(R\langle S\rangle, B) \cong \mathrm{Hom}^{(c)}_{\mathcal{S}\mathrm{ets}}(S, |B|), \quad \textit{functorially.}$$

Sketch of proof. (A better proof *via* tensor algebras will be given later.) Given S, pick a "symbol", X_s, for each $s \in S$, and map $\mathbb{N}$ to the "positive powers of X_s," *via* $n \mapsto X_s^n$, and define $X_s^m \cdot X_s^n = X_s^{m+n}$. Let $\mathbb{N}_s = \{X_s^n \mid n \geq 1\} \cong \mathbb{N}$ (as monoid), and let

$$\mathcal{S} = \coprod_{s \in S} \mathbb{N}_s.$$

Consider $\mathcal{S}^{(p)}$, the cartesian product of p copies of S, with $p \geq 1$. An element of $\mathcal{S}^{(p)}$ is a tuple of the form $(X_{r_1}^{a_1}, \ldots, X_{r_p}^{a_p})$, and is called a *monomial*. Call a monomial *admissible* iff $r_i \neq r_{i+1}$, for $i = 1, \ldots, p-1$. Multiplication of admissible monomials is concatenation, with possible one-step reduction, if necessary. Call $\mathcal{S}^*$ the union of all the admissible monomials from the various $\mathcal{S}^{(p)}$, with $p \geq 1$, together with the "empty monomial", $\emptyset$. Set

$$R\langle S\rangle = \{f\colon \mathcal{S}^* \to R \mid f(\xi) = 0, \quad \text{except for finitely many } \xi \in \mathcal{S}^*\}.$$

There is a map $R \longrightarrow R\langle S\rangle$ ($\alpha \mapsto \alpha\emptyset$). We make $R\langle S\rangle$ into a ring by defining addition and multiplication as in the commutative case:

$$\begin{aligned}(f+g)(\xi) &= f(\xi) + g(\xi)\\ (fg)(\xi) &= \sum_{\substack{\eta,\eta',\\ \eta\eta'=\xi}} f(\eta)g(\eta'),\end{aligned}$$

where ξ, η and η' are admissible monomials. Then, $R\langle S\rangle$ is an R-algebra, and it satisfies Theorem 2.3 (DX). $\square$

Theorem 2.4 *Say T is a subset of S. Then, there exists a canonical injection $i\colon A[T] \to A[S]$, and $A[S]$ becomes an $A[T]$-algebra. In the category of $A[T]$-algebras, we have the isomorphism*

$$A[S] \cong A[T][S - T]$$

(Here $S - T$ denotes the complement of T in S, and A is in CR*.)*

Proof. We have an inclusion, $T \hookrightarrow S$, and for every $B \in \mathrm{CR}^A$, restriction to T gives a surjection

$$\mathrm{res}\colon \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |B|) \longrightarrow \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(T, |B|).$$

Because we are in the category of sets, there is a map, θ, so that $\mathrm{res} \circ \theta = \mathrm{id}$. Now, the maps θ and res induce maps Θ and Res so that $\mathrm{Res} \circ \Theta = \mathrm{id}$, as shown below:

$$\begin{array}{ccc}\mathrm{Hom}_{\mathrm{CR}^A}(A[S], B) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |B|)\\ {\scriptstyle \mathrm{Res}}\downarrow\downarrow{\scriptstyle \Theta} & & {\scriptstyle \mathrm{res}}\downarrow\downarrow{\scriptstyle \theta}\\ \mathrm{Hom}_{\mathrm{CR}^A}(A[T], B) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(T, |B|).\end{array}$$

If we let $B = A[S]$, we get a map $i = \mathrm{Res}(\mathrm{id}_{A[S]})\colon A[T] \longrightarrow A[S]$. If we let $B = A[T]$, then, since Res is onto, there is a map $\pi\colon A[S] \to A[T]$ so that $\mathrm{Res}(\pi) = \mathrm{id}_{A[T]}$. It follows that i is an injection, and thus, $A[S]$ is an $A[T]$-algebra.

We have

$$\mathrm{Hom}_{\mathrm{CR}^{A[T]}}(A[T][S-T], B) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S - T, |B|).$$

The given map, $|A[T]| \longrightarrow |B|$, yields a fixed map, $T \longrightarrow |B|$. For any given map, $S - T \longrightarrow |B|$, therefore, we get a canonical map, $T \amalg (S - T) \longrightarrow |B|$, *i.e.*, $S \longrightarrow |B|$, depending *only* on the map $S - T \longrightarrow |B|$. Therefore, there is an injection

$$\mathrm{Hom}_{\mathrm{CR}^{A[T]}}(A[T][S-T], B) \hookrightarrow \mathrm{Hom}_{\mathrm{CR}^A}(A[S], B),$$

and the image is just $\mathrm{Hom}_{\mathrm{CR}^{A[T]}}(A[S], B)$. By Yoneda's lemma, $A[S] \cong A[T][S-T]$, as an $A[T]$-algebra. $\square$

From now on, we will write $\mathrm{Hom}_A(B, C)$ instead of $\mathrm{Hom}_{\mathrm{CR}^A}(B, C)$ and similarly for RNG^R. If $X^{(\xi)}$ is a monomial, then we set

$$\deg(X^{(\xi)}) = \sum_{s \in S} \xi(s) \in \mathbb{Z}_{\geq 0}.$$

If $f \in A[S]$, say $f = \sum_{(\xi)} a_{(\xi)} X^{(\xi)}$, then

$$\deg(f) = \sup\{\deg(X^{(\xi)}) \mid a_{(\xi)} \neq 0\}.$$

In particular, note that $\deg(0) = -\infty$.

Proposition 2.5 *The canonical map, $A \longrightarrow A[S]$, establishes an isomorphism of A with the polynomials of degree ≤ 0 in $A[S]$. Any $\alpha \neq 0$ in A goes to a polynomial of degree 0, only $0 \in A$ goes to a polynomial of degree < 0. If $f, g \in A[S]$, then*

(a) $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$.

(b) $\deg(fg) \leq \deg(f) + \deg(g)$.
If A is without zero divisors then we have equality in (b) and

(c) The units of $A[S]$ are exactly the units of A.

(d) The ring $A[S]$ has no zero divisors.

Proof. Since we deal with degrees and each of the two polynomials f, g involves finitely many monomials, we may assume that S is a finite set. The map $A \longrightarrow A[S]$ is given by $\alpha \mapsto \alpha \cdot 1$ and 1 has degree 0, so it is trivial that we have an isomorphism of A with the polynomials of degree ≤ 0.

Say $S = \{1, \ldots, n\}$ and label the X_s as $X_1, \ldots, X_n$. The monomials are lexicographically ordered:

$$X_1^{a_1} \cdots X_n^{a_n} < X_1^{b_1} \cdots X_n^{b_n}$$

iff $a_1 = b_1, \ldots, a_j = b_j$ and $a_{j+1} < b_{j+1}$ $(j = 0, \ldots, n-1)$.

(a) If $f = \sum_{(\xi)} a_{(\xi)} X^{(\xi)}$ and $g = \sum_{(\xi)} b_{(\xi)} X^{(\xi)}$, then $f + g = \sum_{(\xi)} (a_{(\xi)} + b_{(\xi)}) X^{(\xi)}$. If $\deg(f+g) > \max\{\deg(f), \deg(g)\}$, then there is some η so that

$$\deg(X^{(\eta)}) > \deg(X^{(\xi)}), \quad \text{for all } \xi \text{ occurring in } f \text{ and } g, \text{ and } a_{(\eta)} + b_{(\eta)} \neq 0,$$

a contradiction.

(b) With f and g as in (a), we have

$$fg = \sum_{\xi} \left(\sum_{\substack{\eta, \eta', \\ \eta\eta' = \xi}} a_{(\eta)} b_{(\eta')} \right) X^{(\xi)}. \tag{$*$}$$

Now,

$$\deg(X^{(\eta)}) + \deg(X^{(\eta')}) = \sum_s (\eta\eta')(s) = \sum_s \xi(s) = \deg(X^{(\xi)}).$$

However, $a_{(\eta)} \neq 0$ implies that $\deg(X^{(\eta)}) \leq \deg(f)$ and $b_{(\eta')} \neq 0$ implies that $\deg(X^{(\eta')}) \leq \deg(g)$, and this shows that $\deg(X^{(\xi)}) \leq \deg(f) + \deg(g)$, for any $X^{(\xi)}$ with nonzero coefficient in $(*)$.

When A is a domain, pick η to be the *first* monomial in the lexicographic ordering with $X^{(\eta)}$ of degree equal to $\deg(f)$, and similarly, pick η' to be the *first* monomial in the lexicographic ordering with $X^{(\eta')}$ of degree equal to $\deg(g)$. Then (DX), $X^{(\eta)}X^{(\eta')}$ is the monomial occurring first in the lexicographic ordering and of degree equal to $\deg(f) + \deg(g)$ in fg. Its coefficient is $a_{(\eta)} b_{(\eta')} \neq 0$, as A has no nonzero divisors; so, we have equality in (b).

(c) Say $u \in A[S]$ is a unit. Then, there is some $v \in A[S]$, so that $uv = vu = 1$. Consequently, $\deg(uv) = 0$, but $\deg(uv) = \deg(u) + \deg(v)$. Thus, $\deg(u) = \deg(v) = 0$ (as $\deg(u), \deg(v) \geq 0$), i.e., u, v are units of A.

(d) If $f, g \neq 0$, then $\deg(fg) = \deg(f) + \deg(g) \geq 0$, so $fg \neq 0$. $\square$

Definition 2.1 Suppose A is a commutative ring and B is a commutative A-algebra. Pick a subset, $S \subseteq |B|$. The set, S, is called *algebraically independent over* A (or *the elements of* S *are independent transcendentals over* A) iff the canonical map, $A[S] \longrightarrow B$, is a monomorphism. The set, S, is *algebraically dependent over* A iff the map, $A[S] \longrightarrow B$, is not a monomorphism. When $S = \{X\}$, then X is *transcendental*, resp. *algebraic over* A iff S is algebraically independent (resp. algebraically dependent) over A. The algebra, B, is a *finitely generated* A*-algebra* iff there is a finite subset, $S \subseteq |B|$, so that the canonical map $A[S] \longrightarrow B$ is surjective.

2.3 Operations on Modules; Finiteness Conditions for Rings and Modules

Let $R \in \mathrm{RNG}$, then by an R-module, we always mean a *left* R-module. Observe that a right R-module is exactly a left R^{op}-module. (Here, R^{op} is the opposite ring, whose multiplication $\cdot_{\mathrm{op}}$ is given by $x \cdot_{\mathrm{op}} y = y \cdot x$.) Every ring, R, is a module over itself and over R^{op}. By ideal, we always mean a *left* ideal. This is just an R-submodule of R. If an ideal, $\mathfrak{I}$, is both a left and a right ideal, then we call $\mathfrak{I}$ a *two-sided ideal*.

Let M be an R-module and $\{M_\alpha\}_{\alpha \in \Lambda}$ be a collection of R-submodules of M.

(0) $\bigcap_\alpha M_\alpha$ is an R-submodule of M.

(1) Note that we have a family of inclusion maps, $M_\alpha \hookrightarrow M$; so, we get an element of $\prod_\alpha \mathrm{Hom}_R(M_\alpha, M)$. But then, we have a map

$$\coprod_{\alpha \in \Lambda} M_\alpha \longrightarrow M. \tag{$*$}$$

We define $\sum_\alpha M_\alpha$, a new submodule of M called *the sum of the* M_α, *via* any of the following three equivalent (DX) ways:

(a) Image of $(\coprod_{\alpha \in \Lambda} M_\alpha \longrightarrow M)$.

(b) $\bigcap\{N \mid (1)\ N \subseteq M$, as R-submodule; (2) $M_\alpha \subseteq N$, for all $\alpha \in \Lambda.\}$

(c) $\{\sum_{\mathrm{finite}} m_\alpha \mid m_\alpha \in M_\alpha\}$.

Clearly, $\sum_\alpha M_\alpha$ is the smallest submodule of M containing all the M_α.

(2) Let S be a subset of M. For any $s \in S$, the map $\rho \mapsto \rho s$, from R to Rs, is a surjection, where $Rs = \{\rho s \mid \rho \in R\}$. Thus, we get the submodule $\sum_{s \in S} Rs$ (equal to the image of $\coprod_S R \longrightarrow M$) and called *the submodule generated by* S; this module is denoted $\mathrm{mod}(S)$ or RS. We say that S *generates* M iff $RS = M$ and that M is a *finitely generated* R*-module* (for short, a *f.g.* R*-module*) iff there is a finite set, S, and a surjection $\coprod_S R \longrightarrow M$.

(3) The *free module on a set,* S*,* is just $\coprod_S R$. Observe that (DX) the functor from $\mathcal{S}\mathrm{ets}$ to $\mathcal{M}\mathrm{od}(R)$ given by $S \rightsquigarrow \coprod_S R$ is the left adjoint of the stripping functor from $\mathcal{M}\mathrm{od}(R)$ to $\mathcal{S}\mathrm{ets}$; i.e., for every R-module, M, we have the functorial isomorphism

$$\mathrm{Hom}_R(\coprod_S R, M) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, |M|).$$

Remark: An R-module, M, is free over R (i.e., $M \cong \coprod_S R$ for some set S) iff M possesses a Hamel basis over R (DX). The basis is indexed by S. To give a homomorphism of a free module to a module, M, is the same as giving the images of a Hamel basis in M, and these images may be chosen arbitrarily.

(4) The transporter of S to N. Let M be an R-module, S be a subset of M and N an R-submodule of M. The *transporter of S to N*, denoted $(S \to N)$, is given by

$$(S \to N) = \{\rho \in R \mid \rho S \subseteq N\}.$$

(Old notation: $(N : S)$. Old terminology: *residual quotient of N by S*.)
When $N = (0)$, then $(S \to (0))$ has a special name: the *annihilator* of S, denoted $\mathrm{Ann}(S)$. Observe:

(a) $(S \to N)$ is *always* an ideal of R.

(b) So, $\mathrm{Ann}(S)$ is an ideal of R. But if S is *a submodule* of M, then $\mathrm{Ann}(S)$ is a two-sided ideal of R. For if $\rho \in \mathrm{Ann}(S)$ and $\xi \in R$, we have $(\rho\xi)(s) = \rho(\xi s) \subseteq \rho S = (0)$. Thus, $\rho\xi \in \mathrm{Ann}(S)$.

(c) Similarly, if S is a *submodule* of M, then $(S \to N)$ is a two-sided ideal of R.

An R-module, M, is *finitely presented* (for short, *f.p.*) iff there are some *finite* sets, S and T, and an exact sequence

$$\coprod_T R \longrightarrow \coprod_S R \longrightarrow M \longrightarrow 0.$$

This means that M is finitely generated and that the kernel, K, of the surjection, $\coprod_S R \longrightarrow M$, is also finitely generated. Note that f.p. implies f.g.

Definition 2.2 An R-module, M, has the *ascending chain condition (ACC)* (resp. the *descending chain condition (DCC))* iff every ascending chain of submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_n \subseteq \cdots,$$

eventually stabilizes (resp. every descending chain of submodules

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \supseteq \cdots,$$

eventually stabilizes.) If M has the ACC it is called *noetherian* and if it has the DCC it is called *artinian*. The module, M, has the *maximal condition* (resp. *minimal condition*) iff every nonempty family of submodules of M has a maximal (resp. minimal) element.

Proposition 2.6 *Given a module, M, over R consider all the statements*

(1) M is noetherian (has the ACC).

(2) M has the maximal property.

(3) Every submodule of M is finitely generated.

(4) M is artinian (has the DCC).

(5) M has the minimal property.

Then, (1)–(3) are equivalent and (4) and (5) are equivalent.

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{F}$ be a given nonempty family of submodules of M. If there is no maximal element of $\mathcal{F}$, given $M_1 \in \mathcal{F}$, there is some M_2 in $\mathcal{F}$ so that $M_1 < M_2$. Repeating the argument, we find there is some $M_3 \in \mathcal{F}$ so that $M_2 < M_3$, and by induction, for every $n \geq 1$, we find some $M_{n+1} \in \mathcal{F}$ so that $M_n < M_{n+1}$. So, we find an infinite strictly ascending chain

$$M_1 < M_2 < M_3 < \cdots < M_t < \cdots,$$

contradicting (1).

(2) $\Longrightarrow$ (3). Observe that the maximal property for M is inherited by every submodule.

Claim: The maximal property for a module implies that it is finitely generated. If so, we are done. Pick M with the maximal property and let

$$\mathcal{F} = \{N \subseteq M \mid N \text{ is a submodule of } N \text{ and } N \text{ is f.g.}\}$$

The family, $\mathcal{F}$, is nonempty since for every $m \in M$, the module $Rm \subseteq M$ is a submodule of M generated by $\{m\}$, and so, $Rm \in \mathcal{F}$. Now, $\mathcal{F}$ has a maximal element, say T. If $T \neq M$, then there is some $m \in M$ with $m \notin T$. But now, $T + Rm > T$ and $T + Rm$ is finitely generated by the generators of T plus the new generator m, a contradiction. Therefore, $M = T \in \mathcal{F}$; and so, M is f.g.

(3) $\Longrightarrow$ (1). Take an ascending chain,

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r \subseteq \cdots,$$

and look at $N = \bigcup_{i=1}^{\infty} M_i$. Note that N is a submodule of M. So, by (3), the module N is finitely generated. Consequently, there is some t so that M_t contains all the generators of N, and then we have $N \subseteq M_t \subseteq M_r \subseteq N$, for all $r \geq t$. Therefore, $M_t = M_r = N$ for all $r \geq t$.

(4) $\Longrightarrow$ (5). The proof is obtained from the proof of (1) $\Longrightarrow$ (2) *mutatis mutandis*.

(5) $\Longrightarrow$ (4). Say

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_r \supseteq \cdots$$

is a descending chain in M. Let $\mathcal{F} = \{M_i \mid i \geq 1\}$. By (5), the family $\mathcal{F}$ has a minimal element, say M_r. Then, it is clear that the chain stabilizes at r. $\square$

Proposition 2.7 *Let M be a module and write (α), (β) and (γ) for the finiteness properties*

(α) *finite generation*

(β) *ACC*

(γ) *DCC*

Then,

(A) If M has any of (α), (β), (γ), so does every factor module of M.

(B) If M has (β) or (γ), so does every submodule of M.

(C) Say $N \subseteq M$ is a submodule and N and M/N have any one of (α), (β), (γ). Then, M also has the same property.

Proof. (A) If M is f.g., then there is a surjection

$$\coprod_S R \longrightarrow M, \quad \text{with } \#(S) \text{ finite.}$$

Let $\overline{M}$ be a factor module of M; there is a surjection $M \longrightarrow \overline{M}$. By composition, we get a surjection

$$\coprod_S R \longrightarrow M \longrightarrow \overline{M},$$

and so, $\overline{M}$ is f.g. Any ascending (resp. descending) chain in $\overline{M}$ lifts to a similar chain of M. The rest is clear.

(B) Any ascending (resp. descending) chain in $N \subseteq M$ is a similar chain of M; the rest is clear.

(C) Say N and M/N have (α). Then, there are two finite (disjoint) sets, S and T, and surjections

$$\coprod_S R \longrightarrow N \longrightarrow 0 \quad \text{and} \quad \coprod_T R \longrightarrow M/N \longrightarrow 0.$$

Consider the diagram:

$$\begin{array}{ccccc}
 & & \coprod_T R & & \\
 & {}^{\theta}\swarrow & \downarrow & & \\
M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow & & \\
 & & 0 & & .
\end{array}$$

As $\coprod_T R$ is free, there exists an arrow, $\theta\colon \coprod_T R \longrightarrow M$, shown above, and the diagram commutes. Now, consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \coprod_S R & \longrightarrow & \coprod_{S\cup T} R & \longrightarrow & \coprod_T R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & & .
\end{array}$$

We obtain the middle vertical arrow by the map θ and the set map $S \longrightarrow M$ (*via* $S \longrightarrow N \hookrightarrow M$). By construction, our diagram commutes. We claim that the middle arrow is surjective. For this, we chase the diagram: Choose m in M and map m to $\overline{m} \in M/N$. There is some $\xi \in \coprod_T R$ so that $\xi \mapsto \overline{m}$. However, ξ comes from $\eta \in \coprod_{S\cup T} R$. Let $\widetilde{\eta}$ be the image in M of η. Since the diagram is commutative, $\overline{\widetilde{\eta}} = \overline{m}$, and so, $\widetilde{\eta} - m$ maps to 0 in M/N. Consequently, there is some $n \in N$ so that $\widetilde{\eta} - m = n$. Yet, n comes from some ρ in $\coprod_S R \hookrightarrow \coprod_{S\cup T} R$ (i.e., $\widetilde{\rho} = n$). Consider $\eta - \rho \in \coprod_{S\cup T} R$. The image of $\eta - \rho$ in M is $\widetilde{\eta} - \widetilde{\rho} = m + n - n = m$, proving surjectivity. As $S \cup T$ is finite, the module, M, has (α).

Next, assume N and M/N have (β). Let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_r \subseteq \cdots$$

be an ascending chain in M. Write $\overline{M}_j$ for the image of M_j in M/N. By the ACC in M/N, there is some $t \geq 1$ so that $\overline{M}_j = \overline{M}_t$ for all $j \geq t$. If we let $N_j = M_j \cap N$, we get an ascending chain in N. By the ACC in N, this chains stabilizes, i.e., there is some $s \geq 1$ so that $N_j = N_s$ for all $j \geq s$. Let $r = \max\{s, t\}$. We claim that $M_j = M_r$ for all $j \geq r$. We have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_r & \longrightarrow & M_r & \longrightarrow & \overline{M}_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_j & \longrightarrow & M_j & \longrightarrow & \overline{M}_j & \longrightarrow & 0,
\end{array}$$

where the rows are exact and the vertical arrows on the left and on the right are surjections. A diagram chase yields the fact that the middle vertical arrow is also surjective.

Finally, assume N and M/N have (γ). The same argument works with the arrows and inclusions reversed.
□

Corollary 2.8 *Say $\{M_\lambda\}_{\lambda\in\Lambda}$ is a family of R-modules. Then, $\coprod_\lambda M_\lambda$ has one of (α), (β), (γ) iff each M_λ has the corresponding property and Λ is finite.*

Proof. We have a surjection
$$\coprod_\lambda M_\lambda \longrightarrow M_\mu \longrightarrow 0, \quad \text{for any fixed } \mu.$$
Consequently, (α), (β), (γ) for $\coprod_\lambda M_\lambda$ implies (α), (β), (γ) for M_μ, by the previous proposition. It remains to prove that Λ is finite.

First, assume that $\coprod_\lambda M_\lambda$ has (α), and further assume that Λ is infinite. There is some finite set, S, and a surjection $\coprod_S R \longrightarrow \coprod_\lambda M_\lambda$. We may assume that $S = \{1, \ldots, n\}$, for some positive integer, n. Then, we have the canonical basis vectors, $e_1, \ldots, e_n$, of $\coprod_S R$, and their images $\overline{e}_1, \ldots, \overline{e}_n$ generate $\coprod_\lambda M_\lambda$. Each image $\overline{e}_i$ is a finite tuple in $\coprod_\lambda M_\lambda$. Yet, the union of the finite index sets so chosen is again finite and for any μ not in this finite set, the image of M_μ in $\coprod_\lambda M_\lambda$ is not covered. This contradicts the fact that the $\overline{e}_i$'s generate $\coprod_\lambda M_\lambda$, and so, Λ must be finite.

We treat (β) and (γ) together. Again, assume that Λ is infinite. Then, there is a countably infinite subset of Λ, denote it $\{\lambda_1, \lambda_2, \ldots\}$, and the chains
$$M_{\lambda_1} < M_{\lambda_1} \amalg M_{\lambda_2} < M_{\lambda_1} \amalg M_{\lambda_2} \amalg M_{\lambda_3} < \cdots$$
and
$$\coprod_{j=1}^{\infty} M_{\lambda_j} > \coprod_{j\neq 1} M_{\lambda_j} > \coprod_{j\neq 1,2} M_{\lambda_j} > \cdots$$
are infinite ascending (resp. descending) chains of $\coprod_\lambda M_\lambda$, a contradiction.

Finally, assume that each M_λ has (α) or (β) or (γ) and that Λ is finite. We use induction on $\#(\Lambda)$. Consider the exact sequence
$$0 \longrightarrow \coprod_{j\neq 1} M_j \longrightarrow \coprod_{j\in\Lambda} M_j \longrightarrow M_1 \longrightarrow 0.$$
Then, (α) (resp. (β), (γ)) holds for the right end by hypothesis, and it also holds for the left end, by induction; so, (α) (resp. (β), (γ)) holds in the middle. □

Corollary 2.9 *Say R is noetherian (has the ACC on ideals) or artinian (has the DCC on ideals). Then,*

(1) Every f.g. free module, $\coprod_S R$, is noetherian, resp. artinian, as R-module (remember, $\#(S) < \infty$).

(2) Every f.g. R-module is noetherian, resp. artinian.

(3) When R is noetherian, every f.g. R-module is f.p. Finitely presented modules are always f.g.

Proof. (1) and (2) are trivial from Corollary 2.8.

As for (3), that f.p. implies f.g. is clear by the definition. Say M is f.g. Then, we have an exact sequence
$$0 \longrightarrow K \longrightarrow \coprod_S R \longrightarrow M \longrightarrow 0,$$
with $\#(S) < \infty$. By (1), the module $\coprod_S R$ is noetherian; by Proposition 2.6, the module K is f.g. Thus, there is some finite set, T, so that
$$\coprod_T R \longrightarrow K \longrightarrow 0 \quad \text{is exact.}$$
By splicing the two sequences, we get the exact sequence
$$\coprod_T R \longrightarrow \coprod_S R \longrightarrow M \longrightarrow 0,$$
which shows that M is f.p. □

Counter-examples.

(1) A subring of a noetherian ring need not be a noetherian ring. Take $A = \mathbb{C}[X_1, X_2, \ldots, X_n, \ldots]$ the polynomial ring in countably many variables, and let $K = \mathrm{Frac}(A)$. Every field is noetherian as a ring (a field only has two ideals, (0) and itself). We have $A \subseteq K$, yet A is not noetherian, for we claim that we have the ascending chain of ideals

$$(X_1) < (X_1, X_2) < (X_1, X_2, X_3) < \cdots$$

Would this chain stabilize, then we would have $(X_1, \ldots, X_n) = (X_1, \ldots, X_n, X_{n+1})$, for some $n \geq 1$. Now, there would be some polynomials $f_1, \ldots, f_n$ in A so that

$$X_{n+1} = f_1X_1 + \cdots + f_nX_n.$$

Map A to $\mathbb{C}$ *via* the unique homomorphism sending X_j to 0 for $j = 1, \ldots, n$, and sending X_j to 1 for $j > n$. We get $1 = 0$, a contradiction. Therefore, the chain is strictly ascending.

(2) A module which is finitely generated need not be finitely presented. Let $A = \mathbb{C}[X_1, \ldots, X_n, \ldots]$, the polynomial algebra over $\mathbb{C}$ in countably many variables. Then, $\mathbb{C}$ is an A-module because of the exact sequence

$$0 \longrightarrow \mathfrak{I} = (X_1, \ldots, X_n, \ldots) \longrightarrow A \longrightarrow \mathbb{C} \longrightarrow 0,$$

in which the map $A \longrightarrow \mathbb{C}$ is given by $f \mapsto f(0)$; the ring A acts on $\mathbb{C}$ *via* $f \cdot z = f(0)z$, where $f \in A$ and $z \in \mathbb{C}$. Assume that $\mathbb{C}$ is finitely presented. Then, there are some finite sets, S and T, and an exact sequence

$$\coprod_T A \longrightarrow \coprod_S A \longrightarrow \mathbb{C} \longrightarrow 0.$$

We get the diagram

$$\begin{array}{ccccccccc}
 & & \coprod_T A & \longrightarrow & \coprod_S A & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow \varphi & & \downarrow \Theta & & \| & & \\
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & A & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

To construct the vertical arrows, let $e_1, \ldots, e_s$ be the usual generators of $\coprod_S A$. If $z_1, \ldots, z_s \in \mathbb{C}$ are their images, there exist $\lambda_1, \ldots, \lambda_s \in A$ so that

$$\sum_{j=1}^{s} \lambda_j e_j \mapsto \sum_{j=1}^{s} \lambda_j(0) z_j = 1.$$

We have the ($\mathbb{C}$-linear) map, $\mathbb{C} \longrightarrow A$, so our z_j lie in A. Then, we have $\sum_{j=1}^{s} \lambda_j(0)z_j = 1$, in A. If we send $e_j \mapsto z_j \in A$, we get an A-linear map, $\Theta\colon \coprod_S A \to A$, and there is some $\xi \in \coprod_S A$ with $\Theta(\xi) = 1 \in A$. Namely, take

$$\xi = \sum_{j=1}^{s} \lambda_j(0) e_j.$$

But then, Θ is onto, because its image is an ideal which contains 1. A diagram chase implies that there exists some $\varphi\colon \coprod_T A \to \mathfrak{I}$ rendering the diagram commutative. Another diagram chase gives the fact that φ is surjective. But then, $\mathfrak{I}$ is finitely generated, a contradiction. Therefore, $\mathbb{C}$ is not f.p. (over A).

Remark: The difficulty is that A is much "bigger" than $\mathbb{C}$, and thus, the surjection $A \longrightarrow \mathbb{C}$ has to "kill" an infinite number of independent elements.

Consider the category, $\mathcal{M}\mathrm{od}(R)$. We can also look at subcategories of $\mathcal{M}\mathrm{od}(R)$ having some additional properties. For example, a subcategory, $\mathcal{C}$, of $\mathcal{M}\mathrm{od}(R)$ is a *localizing subcategory* iff

(a) Whenever M and $N \in \mathcal{O}\mathrm{b}(\mathcal{C})$ and $\theta\colon M \to N$ is a morphism of $\mathcal{C}$, then $\mathrm{Ker}\,\theta$ and $\mathrm{Coker}\,\theta = (N/\mathrm{Im}\,\theta)$ lie in $\mathcal{O}\mathrm{b}(\mathcal{C})$ and the morphisms $\mathrm{Ker}\,\theta \longrightarrow M$ and $N \longrightarrow \mathrm{Coker}\,\theta$ are arrows of $\mathcal{C}$.

(b) Whenever
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \quad \text{is exact (in } \mathcal{M}\mathrm{od}(R))$$
and $M', M'' \in \mathcal{O}\mathrm{b}(\mathcal{C})$, then $M \in \mathcal{O}\mathrm{b}(\mathcal{C})$ and the sequence is exact in $\mathcal{C}$.

Example: Let $\mathcal{C} = \mathcal{M}\mathrm{od}^{\mathrm{fg}}(R)$ be the full subcategory of finitely generated R-modules, where R is noetherian. The reader should check that $\mathcal{C}$ is a localizing subcategory.

Recall that an R-module is a *simple* iff it has *no* nontrivial submodules; a composition series is a finite descending chain
$$M = M_0 > M_1 > M_2 > \cdots > M_t = (0)$$
in which all the factors M_j/M_{j+1} are simple. We know from the Jordan–Hölder theorem that the number of composition factors, t, is an invariant and the composition factors are unique (up to isomorphism and rearrangement). We set $\lambda_R(M) = t$, and call it the *length* of M; if M does not have a composition series, set $\lambda_R(M) = \infty$.

Say $\mathcal{C}$ is a localizing subcategory of $\mathcal{M}\mathrm{od}(R)$ and φ is a function on $\mathcal{O}\mathrm{b}(\mathcal{C})$ to some fixed abelian group, A.

Definition 2.3 The function, φ, is an *Euler function* iff whenever
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \quad \text{is exact in } \mathcal{C},$$
we have $\varphi(M) = \varphi(M') + \varphi(M'')$.

Proposition 2.10 *A necessary and sufficient condition that a module, M, have finite length is that M has both ACC and DCC on submodules. The function λ_R on the full subcategory of finite-length modules (which is a localizing subcategory), is an Euler function. If φ is an Euler function on some localizing subcategory of $\mathcal{M}\mathrm{od}(R)$ and if*
$$(E) \qquad 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_t \longrightarrow 0$$
is an exact sequence in this subcategory, then
$$\chi_\varphi((E)) = \sum_{j=1}^{t} (-1)^j \varphi(M_j) = 0.$$

Proof. First, assume that M has finite length. We prove the ACC and the DCC by induction on $\lambda_R(M)$. If $\lambda_R(M) = 1$, then M is simple, so the ACC and the DCC hold trivially. Assume that this is true for $\lambda_R(M) = t$, and take $\lambda_R(M) = t + 1$. We have a composition series
$$M = M_0 > M_1 > M_2 > \cdots > M_{t+1} = (0),$$
and so, $\lambda_R(M_1) = t$ and $\lambda_R(M/M_1) = 1$. But the sequence
$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0 \quad \text{is exact},$$
and the ACC and DCC hold on the ends, by induction. Therefore, they hold in the middle.

Now, assume that the DCC and the ACC hold for M. Let
$$\mathcal{F} = \{N \subseteq M \mid N \neq M,\ N \text{ is a submodule of } M.\}$$

The family $\mathcal{F}$ is nonempty (the trivial module (0) is in $\mathcal{F}$) and by the ACC, it has a maximal element, M_1; so, M/M_1 is simple. Apply the same argument to M_1: We get $M_2 < M_1$ with M_1/M_2 simple. By induction, we get a strictly descending chain

$$M = M_0 > M_1 > M_2 > \cdots > M_t > \cdots$$

However, by the DCC, this chain must stabilize. Now, if it stabilizes at M_t, we must have $M_t = (0)$, since otherwise we could repeat the first step in the argument for M_t. This proves that $\lambda_R(M) = t < \infty$.

Say $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact in $\mathcal{M}\mathrm{od}^{\mathrm{fl}}(R)$. Pick a composition series for M''. We get a strictly descending chain

$$M'' = M''_0 > M''_1 > M''_2 > \cdots > M''_t = (0).$$

By the second homomorphism theorem, we get a lifted sequence

$$M = M_0 > M_1 > M_2 > \cdots > M_t = M',$$

and if we pick a composition series for M', we get the following composition series with $s + t = \lambda_R(M') + \lambda_R(M'')$ factors, as required:

$$M = M_0 > M_1 > M_2 > \cdots > M_t = M' > M'_1 > M'_2 > \cdots > M'_s = (0).$$

Say

$$(E) \qquad 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{t-2} \longrightarrow M_{t-1} \xrightarrow{\theta} M_t \longrightarrow 0$$

is an exact sequence. Then, we have the two exact sequences

$$\begin{array}{ll} (E') & 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{t-2} \longrightarrow \mathrm{Ker}\ \theta \longrightarrow 0 \quad \text{and} \\ (E'') & 0 \longrightarrow \mathrm{Ker}\ \theta \longrightarrow M_{t-1} \longrightarrow M_t \longrightarrow 0. \end{array}$$

The cases $t = 1, 2, 3$ are trivial (DX). By using induction on t, we see that the proposition is true for (E') and (E''). Thus, we get

$$\begin{aligned} \sum_{j=1}^{t-2} (-1)^j \varphi(M_j) + (-1)^{t-1} \varphi(\mathrm{Ker}\ \theta) &= 0 \quad \text{and} \\ \varphi(\mathrm{Ker}\ \theta) &= \varphi(M_{t-1}) - \varphi(M_t). \end{aligned}$$

If we add the first equation to $(-1)^t$ times the second equation we get

$$\sum_{j=1}^{t-2} (-1)^j \varphi(M_j) = (-1)^t \varphi(M_{t-1}) - (-1)^t \varphi(M_t),$$

and so,

$$\chi_\varphi((E)) = \sum_{j=1}^{t-2} (-1)^j \varphi(M_j) + (-1)^{t-1} \varphi(M_{t-1}) + (-1)^t \varphi(M_t) = 0,$$

as claimed. $\square$

Theorem 2.11 *(Hilbert Basis Theorem (1890)) If A is a commutative noetherian ring, then so is the polynomial ring $A[X]$.*

Proof. Let A_n be the submodule of $A[X]$ consisting of the polynomials of degree at most n. The module, A_n, is a free module over A (for example, $1, X, X^2, \ldots, X^n$ is a basis of A_n). If $\mathfrak{A}$ is an ideal of $A[X]$, then $\mathfrak{A} \cap A_n$ is a submodule of A_n. As A_n (being finitely generated over A) is a noetherian module, $\mathfrak{A} \cap A_n$ is also finitely generated, say by $\alpha_1, \alpha_2, \ldots, \alpha_{\kappa(n)}$ ($\in A[X]$). If $f \in \mathfrak{A}$ and $\deg(f) \leq n$, then $f \in A_n$; so,

$$f = a_1\alpha_1 + \cdots + a_{\kappa(n)}\alpha_{\kappa(n)}, \quad \text{with } a_j \in A.$$

Now, let $\mathfrak{A}^*$ be the subset of A consisting of all $a \in A$ so that either $a = 0$ or there is some polynomial f in $\mathfrak{A}$ having a as its leading coefficient, i.e., $f = aX^r + O(X^{r-1})$. We claim that $\mathfrak{A}^*$ is an ideal of A.

Say a and b are in $\mathfrak{A}^*$. Then, there are some polynomials $f, g \in \mathfrak{A}$ so that $f = aX^r + O(X^{r-1})$ and $g = bX^s + O(X^{s-1})$. Take $t = \max\{r, s\}$. Then, $X^{t-r}f \in \mathfrak{A}$ and $X^{t-s}g \in \mathfrak{A}$, since $\mathfrak{A}$ is an ideal. But,

$$X^{t-r}f = aX^t + O(X^{t-1}) \quad \text{and} \quad X^{t-s}g = bX^t + O(X^{t-1}),$$

and this implies that $a \pm b \in \mathfrak{A}^*$, as $a \pm b$ is the leading coefficient of $X^{t-r}f \pm X^{t-s}g \in \mathfrak{A}$. If $\lambda \in A$ and $a \in \mathfrak{A}^*$, then it is clear that $\lambda a \in \mathfrak{A}^*$. Therefore, $\mathfrak{A}^*$ is indeed an ideal in A. Now, A is a noetherian ring, therefore $\mathfrak{A}^*$ is finitely generated as an ideal. So, there exist $\beta_1, \ldots, \beta_t \in \mathfrak{A}^* \subseteq A$, such that for any $\beta \in \mathfrak{A}^*$, we have $\beta = \sum_{i=1}^t \lambda_i\beta_i$, for some $\lambda_i \in A$. Now, by definition of $\mathfrak{A}^*$, for every $\beta_i \in \mathfrak{A}^*$, there is some $f_i(X) \in \mathfrak{A}$ so that $f_i(X) = \beta_i X^{n_i} + O(X^{n_i-1})$. Let $n = \max\{n_1, \ldots, n_t\}$ and consider the generators $\alpha_1, \ldots, \alpha_{\kappa(n)}$ of $\mathfrak{A}_n = A_n \cap \mathfrak{A}$.

Claim: The set $\{\alpha_1, \ldots, \alpha_{\kappa(n)}, f_1, \ldots, f_t\}$ generates $\mathfrak{A}$.

Pick some $g \in \mathfrak{A}$. Then, $g(X) = \beta X^r + O(X^{r-1})$, for some r. If $r \leq n$, then $g(X) \in \mathfrak{A}_n$, and thus, $g = \lambda_1\alpha_1 + \cdots + \lambda_{\kappa(n)}\alpha_{\kappa(n)}$, with $\lambda_i \in A$. Say $r > n$. Now, $\beta \in \mathfrak{A}^*$, so there are elements $\lambda_1, \ldots, \lambda_t \in A$ such that $\beta = \lambda_1\beta_1 + \cdots + \lambda_t\beta_t$. Consider the polynomial

$$P(X) = \sum_{i=1}^t \lambda_i X^{r-n_i} f_i(X),$$

and examine $g(X) - P(X)$. We have

$$g(X) - P(X) = \beta X^r - \sum_{i=1}^t \lambda_i X^{r-n_i} f_i(X) + O(X^{r-1}) = O(X^{r-1}),$$

and thus there is a $P(X) \in (f_1, \ldots, f_t)$ so that $\deg(g(X) - P(X)) \leq r - 1$. By repeating this process, after finitely many steps, we get

$$g(X) - \sum_{i=1}^t h_i(X) f_i(X) = O(X^{\leq n}).$$

Since this polynomial belongs to $\mathfrak{A}$, we deduce that it belongs to $\mathfrak{A}_n$. However, $\mathfrak{A}_n$ is generated by $\alpha_1, \ldots, \alpha_{\kappa(n)}$, and so, $g(X)$ is an $A[X]$-linear combination of the $f_i(X)$'s and the $\alpha_j(X)$'s, as desired. □

Remark: The reader should reprove Hilbert's theorem using the same argument but involving ascending chains. This is Noether's argument (DX).

Corollary 2.12 *Say $R \in$ RNG. If R is noetherian, so is $R\langle X \rangle$.*

Proof. We have $R\langle X \rangle = R[X]$, and the same proof works. □

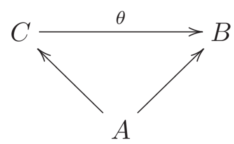
Corollary 2.13 *If A (in* CR*) is noetherian, then so is $A[X_1, \ldots, X_n]$.*

Corollary 2.14 *(Hilbert's original theorem) The polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ is noetherian. If k is a field (Hilbert chose $\mathbb{C}$) then $k[X_1, \ldots, X_n]$ is noetherian.*

Corollary 2.15 *(of the proof–(DX)) If k is a field, then $k[X]$ is a PID.*

Corollary 2.16 *Say A is a noetherian ring ($A \in \mathrm{CR}$) and B is a finitely generated A-algebra. Then, B is a noetherian ring.*

Proof. The hypothesis means that B is a homomorphic image of a polynomial ring $C = A[X_1, \ldots, X_n]$ in such a way that the diagram



commutes, where $A \longrightarrow C$ is the natural injection of A into $C = A[X_1, \ldots, X_n]$. The ring $A[X_1, \ldots, X_n]$ is noetherian, by Corollary 2.13. The map θ makes B into a C-module and B is finitely generated as *C-module*. Now, C-submodules are exactly the ideals of B (DX). Since B is finitely generated as C-module and C is noetherian, this implies that B is a noetherian C-module. Therefore, the ACC on C-submodules holds, and since these are ideals of B, the ring B is noetherian. □

To be finitely generated as A-algebra is **very** different from being finitely generated as A-module.

Given an exact sequence of modules,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

there are situations where it is useful to know that M' is f.g, given that M and M'' satisfy certain finiteness conditions. We will give below a proposition to this effect. The proof makes use of Schanuel's lemma. First, introduce the following terminology: Given a module M, call an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

a *presentation* of M if F is free. Note that M is f.p. iff there is a presentation of M in which both F and K are f.g.

Proposition 2.17 *If M is a Λ-module, then M is f.p. iff every presentation*

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0, \qquad (*)$$

in which F is f.g. has K f.g. and at least one such exists.

Proof. The direction ($\Leftarrow$) is clear.

($\Rightarrow$). Say M is f.p.; we have an exact sequence

$$0 \longrightarrow K' \longrightarrow F' \longrightarrow M \longrightarrow 0,$$

where both K' and F' are f.g. and F' is free. Pick any presentation, $(*)$, with F f.g. If we apply Schanuel's lemma, we get

$$F' \coprod K \cong F \coprod K',$$

But, the righthand side is f.g. and K is a quotient of the left hand side, so it must be f.g. □

Remark: The forward implication of Proposition 2.17 also holds even if F is not free. A simple proof using the snake lemma will be given at the end of Section 2.5.

2.4 Projective and Injective Modules

Let $F\colon \mathcal{M}\mathrm{od}(R) \to \mathcal{M}\mathrm{od}(S)$ be a functor (where $R, S \in \mathrm{RNG}$). Say

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \qquad (*)$$

is exact in $\mathcal{M}\mathrm{od}(R)$. What about

$$0 \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0? \qquad (**)$$

(1) The sequence $(**)$ is a complex if F is an *additive* functor. (Observe that $\mathrm{Hom}_R(M,N)$ is an abelian group, so is $\mathrm{Hom}_S(F(M),F(N))$. We say F is additive iff $\mathrm{Hom}_R(M,N) \xrightarrow{F(\cdot)} \mathrm{Hom}_S(F(M),F(N))$ is a homomorphism, i.e., preserves addition.)

(2) The functor, F, is *exact* iff when $(*)$ is exact then $(**)$ is exact (the definition for cofunctors is identical).

(3) The functor, F, is a *left-exact* (resp. *right-exact*) iff when $(*)$ is exact

$$0 \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \qquad (***)$$

is still exact (resp.

$$F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0 \qquad (****)$$

is still exact.)

(4) The functor, F, is *half-exact* (same definition for cofunctors) iff when $*$ is exact

$$F(M') \longrightarrow F(M) \longrightarrow F(M'')$$

is still exact.

(5) The cofunctor, G, is *left-exact* (resp. *right-exact*) iff $(*)$-exact implies

$$0 \longrightarrow G(M'') \longrightarrow G(M) \longrightarrow G(M')$$

is still exact (resp.

$$G(M'') \longrightarrow G(M) \longrightarrow G(M') \longrightarrow 0$$

is still exact.)

Remark: *The chirality of a functor is determined by the image category.*

Examples of exact (left-exact, right-exact, etc.) functors:

(1) Let $F\colon \mathcal{M}\mathrm{od}(R) \to \mathcal{M}\mathrm{od}(\mathbb{Z})$ be given by: $F(M) =$ underlying abelian group of M. The functor F is exact.

(2) Take a set, Λ, and look at

$$\mathcal{M}\mathrm{od}(R)^\Lambda = \{\{M_\alpha\}_{\alpha\in\Lambda} \mid \text{each } M_\alpha \in \mathcal{M}\mathrm{od}(R)\},$$

together with obvious morphisms. We have two functors from $\mathcal{M}\mathrm{od}(R)^\Lambda$ to $\mathcal{M}\mathrm{od}(R)$. They are:

$$\{M_\alpha\} \rightsquigarrow \prod_\alpha M_\alpha \quad \text{and} \quad \{M_\alpha\} \rightsquigarrow \coprod_\alpha M_\alpha.$$

Both are exact functors (this is special to modules). The next proposition is a most important example of left-exact functors:

Proposition 2.18 *Fix an R-module, N. The functor from $\mathcal{M}\mathrm{od}(R)$ to $\mathcal{A}\mathrm{b}$ (resp. cofunctor from $\mathcal{M}\mathrm{od}(R)$ to $\mathcal{A}\mathrm{b}$) given by $M \rightsquigarrow \mathrm{Hom}_R(N, M)$ (resp. $M \rightsquigarrow \mathrm{Hom}_R(M, N)$) is left-exact (***N.B.: both are left-exact***).*

Proof. Consider the case of a cofunctor (the case of a functor is left to the reader (DX)). Assume that

$$0 \longrightarrow M' \stackrel{\varphi}{\longrightarrow} M \stackrel{\psi}{\longrightarrow} M'' \longrightarrow 0$$

is exact. Look at the sequence obtained by applying $\mathrm{Hom}_R(-, N)$ to the above exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \stackrel{\Psi}{\longrightarrow} \mathrm{Hom}_R(M, N) \stackrel{\Phi}{\longrightarrow} \mathrm{Hom}_R(M', N) \longrightarrow 0,$$

where $\Phi = -\circ\varphi$ and $\Psi = -\circ\psi$. Pick $\alpha \in \mathrm{Hom}_R(M'', N)$ and assume that $\Psi(\alpha) = 0$. We have the commutative diagram

$$\begin{array}{ccc} M & \stackrel{\psi}{\longrightarrow} & M'' \\ & {\scriptstyle\Psi(\alpha)}\searrow & \downarrow{\scriptstyle\alpha} \\ & & N \end{array}$$

and since $M \stackrel{\psi}{\longrightarrow} M''$ is surjective, we deduce that $\alpha = 0$. Now, pick $\beta \in \mathrm{Hom}_R(M, N)$ and assume that $\Phi(\beta) = 0$. We have the commutative diagram (see argument below)

$$\begin{array}{ccccc} M' & \stackrel{\varphi}{\longrightarrow} & M & \stackrel{\psi}{\longrightarrow} & M'' \\ & {\scriptstyle\Phi(\beta)}\searrow & \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\overline{\beta}} & \\ & & N & & \end{array}.$$

Since $\Phi(\beta) = 0$, we have $\mathrm{Im}\ \varphi \subseteq \mathrm{Ker}\ \beta$; so, by the first homomorphism theorem, there is a homomorphism $\overline{\beta}\colon M/M' = M'' \to N$, as shown, making the above diagram commute. Thus, $\Psi(\overline{\beta}) = \overline{\beta} \circ \psi = \beta$, and so, $\beta \in \mathrm{Im}\ \Psi$. □

There may be some modules, N, so that our Hom functors become exact as functors of M. This is the case for the class of R-modules introduced in the next definition:

Definition 2.4 A module, P, is *projective (over R)* iff the functor $M \rightsquigarrow \mathrm{Hom}_R(P, M)$ is exact. A module, Q, is *injective (over R)* iff the cofunctor $M \rightsquigarrow \mathrm{Hom}_R(M, Q)$ is exact.

Remarks:

(1) Any free R-module is projective over R.

Proof. Say $F = \coprod_S R$. Consider the functor $M \rightsquigarrow \mathrm{Hom}_R(\coprod_S R, M)$. The righthand side is equal to $\prod_S \mathrm{Hom}_R(R, M) = \prod_S M$, but we know that the functor $M \rightsquigarrow \prod_S M$ is exact. □

(2) A functor is left-exact iff it preserves the left-exactness of a short left-exact sequence (resp. a cofunctor is left-exact iff it transforms a short right-exact sequence into a left-exact sequence), and *mutatis mutandis* for right exact functors or cofunctors.

(3) Compositions of left (resp. right) exact functors are left (resp. right) exact. Similarly, compositions of exact functors are exact.

We say that an exact sequence

$$0 \longrightarrow M' \stackrel{i}{\longrightarrow} M \stackrel{p}{\longrightarrow} M'' \longrightarrow 0$$

splits iff there is a map $\sigma\colon M'' \to M$ so that $p \circ \sigma = \mathrm{id}_{M''}$. Such a map, σ, is called a *splitting* of the sequence. The following properties are equivalent (DX):

Proposition 2.19 *(1) The sequence*

$$0 \longrightarrow M' \xrightarrow{\;i\;} M \xrightarrow{\;p\;} M'' \longrightarrow 0$$

splits.

(2) Given our sequence as in (1),

$$0 \longrightarrow M' \xrightarrow{\;i\;} M \xrightarrow{\;p\;} M'' \longrightarrow 0$$

there is a map $\pi\colon M \to M'$ *so that* $\pi \circ i = \mathrm{id}_{M'}$.

(3) There is an isomorphism $M' \amalg M'' \cong M$.

Proposition 2.20 *Let* P *be an* R*-module, then the following are equivalent:*

(1) P *is projective over* R.

(2) Given a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \xi} & & \\ M & \longrightarrow & M'' & \longrightarrow & 0, \end{array}$$

there exists a map, $\theta\colon P \to M$, *lifting* ξ, *rendering the diagram commutative (lifting property).*

(3) Any exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow P \longrightarrow 0$ *splits.*

(4) There exists a free module, F, *and another module,* $\widetilde{P}$, *so that* $P \amalg \widetilde{P} \cong F$.

Proof. (1) ⇒ (2). Given the projective module, P and the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \xi} & & \\ M & \longrightarrow & M'' & \longrightarrow & 0, \end{array}$$

the exact sequence gives the map

$$\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, M'') \qquad (\dagger)$$

and the diagram gives an element, ξ, of $\mathrm{Hom}_R(P, M'')$. But P is projective, and so, (†) is surjective. Consequently, ξ comes from some $\eta \in \mathrm{Hom}_R(P, M)$, proving the lifting property.

(2) ⇒ (3). Given an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow P \longrightarrow 0,$$

we get the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \Vert & & \\ M & \longrightarrow & P & \longrightarrow & 0. \end{array}$$

The lifting property gives the backwards map $P \longrightarrow M$, as required.

(3) ⇒ (4). Given P, there is a free module, F. and a surjection, $F \longrightarrow P$. We get the exact sequence

$$0 \longrightarrow \widetilde{P} \longrightarrow F \longrightarrow P \longrightarrow 0,$$

where $\widetilde{P} = \mathrm{Ker}\,(F \longrightarrow P)$. By hypothesis, this sequence splits. Therefore, by property (3) of Proposition 2.19, we have $F \cong P \amalg \widetilde{P}$.

(4) $\Rightarrow$ (1). We have $F \cong P \amalg \widetilde{P}$, for some free R-module, F. Now, $F = \coprod_S R$, for some set, S, and so, for any N,

$$\mathrm{Hom}_R(F, N) = \prod_S \mathrm{Hom}_R(R, N) = \prod_S N.$$

The functor $N \rightsquigarrow \mathrm{Hom}_R(F, N)$ is exact; yet, this functor is $N \rightsquigarrow \mathrm{Hom}_R(P, N) \prod \mathrm{Hom}_R(\widetilde{P}, N)$. Assume that the sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \quad \text{is exact,}$$

we need to show that $\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, M'')$ is surjective. This follows by chasing the diagram (DX):

$$\begin{array}{ccc}
\mathrm{Hom}(F, M) & \xrightarrow{\cong} & \mathrm{Hom}(P, M) \prod \mathrm{Hom}(\widetilde{P}, M) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(F, M'') & \xrightarrow{\cong} & \mathrm{Hom}(P, M'') \prod \mathrm{Hom}(\widetilde{P}, M'') \\
\downarrow & & \\
0 & & .
\end{array}$$

□

Given an R-module, M, a *projective resolution* (resp. a *free resolution*) of M is an exact (possibly infinite) sequence (= *acyclic resolution*) of modules

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M,$$

with all the $P_i's$ projective (resp. free)

Corollary 2.21 *Every R-module possesses a projective resolution (even a free resolution).*

Proof. Since free modules are projective, it is enough to show that free resolutions exist. Find a free module, F_0, so that there is a surjection, $F_0 \longrightarrow M$. Let $M_1 = \mathrm{Ker}\,(F_0 \longrightarrow M)$, and repeat the process. We get a free module, F_1, and a surjection, $F_1 \longrightarrow M_1$. By splicing the two exact sequences $0 \longrightarrow M_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ and $F_1 \longrightarrow M_1 \longrightarrow 0$, we get the exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$. We obtain a free resolution by repeating the above process. □

Proposition 2.22 *Given a family, $\{P_\alpha\}_{\alpha \in \Lambda}$, of modules, the coproduct $\coprod_\alpha P_\alpha$ is projective iff each P_α is projective.*

Proof. Assume that each P_α is projective. This means that for every α, the functor $M \rightsquigarrow \mathrm{Hom}_R(P_\alpha, M)$ is exact. As the product functor is exact and composition of exact functors is exact, the functor $M \rightsquigarrow \prod_\alpha \mathrm{Hom}_R(P_\alpha, M)$ is exact. But

$$\prod_\alpha \mathrm{Hom}_R(P_\alpha, M) = \mathrm{Hom}_R(\coprod_\alpha P_\alpha, M).$$

Therefore, $\coprod_\alpha P_\alpha$ is projective.

Conversely, assume that $\coprod_\alpha P_\alpha$ is projective. By Proposition 2.20, there is a free module, F, and another (projective) module, $\widetilde{P}$, with

$$\Big(\coprod_\alpha P_\alpha\Big) \coprod \widetilde{P} \cong F.$$

Pick any β, then

$$P_\beta \coprod \left(\Big(\coprod_{\alpha \neq \beta} P_\alpha \Big) \coprod \widetilde{P} \right) \cong F.$$

Again, by Proposition 2.20, the module P_β is projective. $\square$

The product of projectives *need not* be projective. (See, HW Problem V.B.VI.)

Remark: Projective modules can be viewed as a natural generalization of free modules. The following characterization of projective modules in terms of linear forms is an another illustration of this fact. Moreover, this proposition can used to prove that invertible ideals of an integral domain are precisely the projective ideals, a fact that plays an important role in the theory of Dedekind rings (see Chapter 3, Section 3.6).

Proposition 2.23 *An R-module, M, is projective iff there exists a family, $\{e_i\}_{i\in I}$, of elements of M and a family, $\{\varphi_i \colon M \to R\}_{i\in I}$, of R-linear maps such that*

(i) For all $m \in M$, we have $\varphi_i(m) = 0$, for all but finitely $i \in I$.

(ii) For all $m \in M$, we have $m = \sum_i \varphi_i(m) e_i$.

In particular, M is generated by the family $\{e_i\}_{i\in I}$.

Proof. First, assume that M is projective and let $\psi \colon F \to M$ be a surjection from a free R-module, F. The map, ψ, splits, we let $\varphi \colon M \to F$ be its splitting. If $\{f_i\}_{i\in I}$ is a basis of F, we set $e_i = \psi(f_i)$. Now, for each $m \in M$, the element $\varphi(m)$ can be written uniquely as

$$\varphi(m) = \sum_k r_k f_k,$$

where $r_k \in R$ and $r_k = 0$ for all but finitely many k. Define $\varphi_i \colon M \to R$ by $\varphi_i(m) = r_i$; it is clear that φ_i is R-linear and that (i) holds. For every $m \in M$, we have

$$m = (\psi \circ \varphi)(m) = \psi\Big(\sum_k r_k f_k \Big) = \sum_k \varphi_k(m) e_k,$$

which is (ii). Of course, this also shows the e_k generate M.

Conversely, assume (i) and (ii). Consider the free module $F = \coprod_{i\in I} R$ and let $\{f_i\}_{i\in I}$ be a basis of F. Define the map $\psi \colon F \to M$ *via* $f_i \mapsto e_i$. To prove that M is projective, by Proposition 2.20 (4), it is enough to find a map $\varphi \colon M \to F$ with $\psi \circ \varphi = 1_M$. Define φ *via*

$$\varphi(m) = \sum_k \varphi_k(m) f_k.$$

The sum on the righthand side is well-defined because of (i), and by (ii),

$$(\psi \circ \varphi)(m) = \sum_k \varphi_k(m) e_k = m.$$

Therefore, M is a cofactor of a free module, so it is projective. $\square$

We would like to test submodules, L, of M as to whether $L = M$ by testing *via* surjections $M \longrightarrow N$. That is, suppose we know that for every N and every surjection $M \longrightarrow N$ we have $L \hookrightarrow M \longrightarrow N$ is also surjective. How restrictive can we be with the N's, yet get a viable test?

There may be some superfluous N, e.g., those N for which $M \longrightarrow N \longrightarrow 0$ automatically implies that $L \longrightarrow M \longrightarrow N$ is surjective. There may even be some such N's that work for *all* L. Thus, it is preferable to fix attention on N and seek small enough M so that N matters in the testing of all L. This yields a piece of the following definition:

Definition 2.5 A surjection, $M \longrightarrow N$, is a *minimal* (*essential*, or *covering*) *surjection* iff for all $L \subseteq M$, whenever $L \longrightarrow M \longrightarrow N$ is surjective, we can conclude $L = M$. A submodule, K, is *small* (*superfluous*) iff for every submodule, $L \subseteq M$, when $L + K = M$, then $L = M$. A submodule, K, is *large* (*essential*) iff for all submodules, $L \subseteq M$, when $L \cap K = (0)$, then $L = (0)$. The injection $K \longrightarrow M$ is *essential* (*minimal*) iff K is large.

Proposition 2.24 *The following are equivalent for surjections $\theta\colon M \to N$:*

(1) $M \stackrel{\theta}{\longrightarrow} N$ *is a minimal surjection.*

(2) $\mathrm{Ker}\ \theta$ *is small.*

(3) $\mathrm{Coker}\,(L \longrightarrow M \longrightarrow N) = (0)$ *implies* $\mathrm{Coker}\,(L \longrightarrow M) = (0)$*, for any submodule,* $L \subseteq M$*.*

Proof. (1) $\Rightarrow$ (2). Pick L, and assume $L + \mathrm{Ker}\ \theta = M$. So, $\theta(L) = \theta(M) = N$. Thus, $L = M$, by (1), which shows that $\mathrm{Ker}\ \theta$ is small.

(2) $\Rightarrow$ (3). Say $L \subseteq M$ and assume that $\mathrm{Coker}\,(L \longrightarrow M \longrightarrow N) = (0)$. Therefore, $N = \mathrm{Im}\,(L \longrightarrow N)$, and we deduce that

$$L + \mathrm{Ker}\ \theta = M,$$

by the second homomorphism theorem. By (2), we get $L = M$; so, $\mathrm{Coker}\,(L \longrightarrow M) = (0)$.

(3) $\Rightarrow$ (1). This is just the definition. □

Definition 2.6 A surjection $P \longrightarrow N$ is a *projective cover* iff

(1) The module P is projective

(2) It is a minimal surjection.

Projective covers may not exist. For example, $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjection and $\mathbb{Z}$ is projective. If $P \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is a projective cover, then the lemma below implies that P is torsion-free. Hence, we can replace $P \longrightarrow \mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. However, the following argument now shows that $\mathbb{Z}/2\mathbb{Z}$ has no projective cover. We have the surjection $\theta\colon \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. This is not a minimal surjection because $2\mathbb{Z}$ is not small. (Clearly, $\mathrm{Ker}\ \theta = 2\mathbb{Z}$; so, say $L = d\mathbb{Z}$ and $d\mathbb{Z} + 2\mathbb{Z} = \mathbb{Z}$. Then, $(d, 2) = 1$, so d is odd. Yet, $d\mathbb{Z} = \mathbb{Z}$ only when $d = 1$. Thus, the module $2\mathbb{Z}$ is not small.) Now, suppose $d\mathbb{Z} \stackrel{\theta}{\longrightarrow} \mathbb{Z}/2\mathbb{Z}$ is surjective, then d must be odd. If $k\mathbb{Z} \subseteq d\mathbb{Z}$ maps onto $\mathbb{Z}/2\mathbb{Z}$, then, as $\mathrm{Ker}\ \theta = 2d\mathbb{Z}$, we get $(k, 2d) = d$. Let $b = k/d$; the integer b must be odd. Then, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & 2d\mathbb{Z} & \longrightarrow & d\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0,
\end{array}$$

(in which the vertical arrows are isomorphisms: multiply by d) shows that the inclusion $k\mathbb{Z} \subseteq d\mathbb{Z}$ corresponds to the inclusion $b\mathbb{Z} \subseteq \mathbb{Z}$. Our previous argument implies $b = 1$; so, $k = d$, and $d\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is not minimal.

Lemma 2.25 *If R has no zero divisors and P is a projective R-module then P is torsion-free.*

Proof. Since the torsion-free property is inherited by submodules, we may assume that P is a free module. Moreover, coproducts of torsion-free modules are torsion-free, so we may assume that $P = R$. But, R has no zero-divisors; so, it is torsion-free. □

Proposition 2.26 *Say R is a ring and $\mathcal{J}(R)$ is its Jacobson radical (i.e., $\mathcal{J}(R)$ is equal to the intersection of all maximal ideals of R). Then, the surjection $R \longrightarrow R/\mathcal{J}(R)$ is a projective cover. In particular, when R is commutative local, then $R \longrightarrow R/\mathfrak{m}_R$ is a projective cover.*

Proof. Pick $L \subseteq R$, a submodule of R, i.e., an ideal of R, such that $L + \mathcal{J}(R) = R$. If $L \neq R$, then $L \subseteq \mathfrak{M}$, where $\mathfrak{M}$ is some maximal ideal. But, $\mathcal{J}(R) \subseteq \mathfrak{M}$, and so $L + \mathcal{J}(R) \subseteq \mathfrak{M}$. The latter inclusion shows that $L + \mathcal{J}(R) \neq R$, a contradiction; so, $\mathcal{J}(R)$ is small. □

For injective modules, the situation is nearly dual to the projective case. It is exactly dual as far as categorical properties are concerned. However, the notion of free module is not categorical, and so, results about projective modules involving free modules have no counterpart for injective modules. On the other hand, the situation for injectives is a bit better than for projectives.

Proposition 2.27 *The following are equivalent for a module, Q:*

(1) The module, Q, is injective.

(2) Given a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \longrightarrow & M \\ & & \big\downarrow{\scriptstyle \xi} & & \\ & & Q & & \end{array}$$

there exists an extension, $\theta\colon M \to Q$, of ξ, making the diagram commute (extension property).

(3) Every exact sequence $0 \longrightarrow Q \longrightarrow M \longrightarrow M'' \longrightarrow 0$ splits.

Proof. (DX)

Proposition 2.28 *Given a family, $\{Q_\alpha\}_{\alpha\in\Lambda}$, of modules, the product $\prod_\alpha Q_\alpha$ is injective iff each Q_α is injective.*

Proof. (DX)

Theorem 2.29 *(Baer Representation Theorem) An R-module, Q, is injective iff it has the extension property w.r.t. the sequence*

$$0 \longrightarrow \mathfrak{A} \longrightarrow R, \qquad (*)$$

where $\mathfrak{A}$ is an ideal of R.

Proof. If Q is injective, it is clear that Q has the extension property w.r.t. $(*)$.

Conversely, assume that the extension property holds for $(*)$. What does this mean? We have the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathfrak{A} & \longrightarrow & R \\ & & {\scriptstyle \varphi}\big\downarrow & \swarrow{\scriptstyle \psi} & \\ & & Q & & \end{array}$$

in which ψ extends φ; so, for all $\xi \in \mathfrak{A}$, we have $\varphi(\xi) = (\psi \restriction \mathfrak{A})(\xi)$. In particular, $\psi(1) \in Q$ exists, say $q = \psi(1)$. Since $\xi \cdot 1 = \xi$ for all $\xi \in \mathfrak{A}$, we have

$$\varphi(\xi) = \psi(\xi) = \xi\psi(1) = \xi q.$$

Given the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \longrightarrow & M \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ & & Q & & \end{array}$$

define $\mathcal{S}$ by

$$\mathcal{S} = \left\{ (N, \psi) \;\middle|\; \begin{array}{l} (1)\ N \text{ is a submodule of } M, \quad (2)\ M' \subseteq N, \\ (3)\ \psi\colon N \to Q \text{ extends } \varphi \text{ to } N. \end{array} \right\}$$

Partially order $\mathcal{S}$ by inclusion and agreement of extensions. Then, $\mathcal{S}$ is inductive (DX). By Zorn's lemma, there is a maximal element, (N_0, ψ_0), in $\mathcal{S}$. We claim that $N_0 = M$. If $N_0 \neq M$, there is some $m \in M - N_0$. Let $\mathfrak{A}$ be the transporter of m into N_0, i.e.,

$$(m \longrightarrow N_0) = \{\rho \in R \mid \rho m \in N_0\}.$$

Define the R-module map, $\theta\colon \mathfrak{A} \to Q$, by $\theta(\rho) = \psi_0(\rho m)$. Look at the module $N_0 + Rm$, which strictly contains N_0. If $z \in N_0 + Rm$, then $z = z_0 + \rho m$, for some $z_0 \in N_0$ and some $\rho \in R$. Set

$$\psi(z) = \psi_0(z_0) + \rho q,$$

where $q = \Theta(1)$ and Θ is an extension of θ (guaranteed to exist, by the hypothesis). We must prove that ψ is a well-defined map, i.e., if $z = z_0 + \rho m = \widetilde{z_0} + \widetilde{\rho} m$, then

$$\psi_0(z_0) + \rho q = \psi_0(\widetilde{z_0}) + \widetilde{\rho} q.$$

Now, if $\psi\colon N_0 + Rm \to Q$ is indeed well-defined, then it is an extension of ψ_0 to the new module $N_0 + Rm > N_0$, contradicting the maximality of N_0. Therefore, $N_0 = M$, and we are done.

If $z = z_0 + \rho m = \widetilde{z_0} + \widetilde{\rho} m$, then $z_0 - \widetilde{z_0} = (\widetilde{\rho} - \rho)m$; so $\widetilde{\rho} - \rho \in \mathfrak{A}$. Consequently,

$$\theta(\widetilde{\rho} - \rho) = \psi_0((\widetilde{\rho} - \rho)m).$$

Yet,

$$\theta(\widetilde{\rho} - \rho) = \Theta(\widetilde{\rho} - \rho) = (\widetilde{\rho} - \rho)\Theta(1) = (\widetilde{\rho} - \rho)q,$$

and so, we get

$$\psi_0(z_0 - \widetilde{z_0}) = \psi_0((\widetilde{\rho} - \rho)m) = \theta(\widetilde{\rho} - \rho) = (\widetilde{\rho} - \rho)q.$$

Therefore, we deduce that

$$\psi_0(z_0) + \rho q = \psi_0(\widetilde{z_0}) + \widetilde{\rho} q,$$

establishing that ψ is well-defined. □

Recall that an R-module, M, is *divisible* iff for every $\lambda \in R$ with $\lambda \neq 0$, the map $M \xrightarrow{\lambda} M$ (multiplication by λ), is surjective.

Corollary 2.30 *If $R \in$ CR has no zero-divisors, then an injective R-module is automatically divisible. Moreover, if R is a P.I.D., a necessary and sufficient condition that Q be injective is that Q be divisible. Therefore, over P.I.D.'s, every factor module of an injective is injective.*

Proof. Let $\lambda \in R$, with $\lambda \neq 0$. Since R has no zero divisors, the map $R \xrightarrow{\lambda} R$ is a monomorphism. Thus, the image of this map is an ideal, $\mathfrak{A}$, and the exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow R$$

is just the exact sequence

$$0 \longrightarrow R \stackrel{\lambda}{\longrightarrow} R.$$

Apply the cofunctor $\mathrm{Hom}_R(-, Q)$. If Q is injective, this cofunctor is exact, and we get the exact sequence

$$\mathrm{Hom}_R(R, Q) \stackrel{\lambda}{\longrightarrow} \mathrm{Hom}_R(R, Q) \longrightarrow 0.$$

So, the sequence $Q \stackrel{\lambda}{\longrightarrow} Q \longrightarrow 0$ is exact, which proves that Q is divisible.

If R is a P.I.D., then every ideal is principal, so, every exact sequence $0 \longrightarrow \mathfrak{A} \longrightarrow R$, where $\mathfrak{A}$ is an ideal, is of the form $0 \longrightarrow R \stackrel{\lambda}{\longrightarrow} R$, for some $\lambda \in R$. If Q is divisible, the sequence $Q \stackrel{\lambda}{\longrightarrow} Q \longrightarrow 0$ is exact, and we get that

$$\mathrm{Hom}_R(R, Q) \stackrel{\lambda}{\longrightarrow} \mathrm{Hom}_R(R, Q) \longrightarrow 0 \quad \text{is exact;}$$

this means that $\mathrm{Hom}_R(-, Q)$ is exact on sequences

$$0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow R/\mathfrak{A} \longrightarrow 0,$$

where $\mathfrak{A}$ is an ideal, i.e., the extension property holds for ideals, $\mathfrak{A}$, of R. By applying Baer's theorem we conclude that Q is injective.

The reader will easily verify that factor modules of divisible modules are divisible (DX). Consequently, the last statement of the corollary holds. $\square$

Theorem 2.31 *(Baer Embedding Theorem) Every R-module is a submodule of an injective module.*

Proof. The proof assigned for homework (Problem 57) is based on Eckmann's proof. Here is Godement's proof [18] (probably the shortest proof). The first step is to show that any $\mathbb{Z}$-module, M, can be embedded into M^{DD}, where $M^D = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Given a $\mathbb{Z}$-module, M, we define a natural $\mathbb{Z}$-linear map, $m \mapsto \widehat{m}$, from M to M^{DD}, in the usual way: For every $m \in M$ and every $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$,

$$\widehat{m}(f) = f(m).$$

Proposition 2.32 *For every $\mathbb{Z}$-module, M, the natural map $M \longrightarrow M^{DD}$ is injective.*

Proof. It is enough to show that $m \neq 0$ implies that there is some $f \in M^D = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ so that $f(m) \neq 0$.

Consider the cyclic subgroup, $\mathbb{Z}m$, of M, generated by m. We define a $\mathbb{Z}$-linear map, $f\colon \mathbb{Z}m \to \mathbb{Q}/\mathbb{Z}$, as follows: If m has infinite order, let $f(m) = 1/2\,(\mathrm{mod}\,\mathbb{Z})$; if m has finite order, n, let $f(m) = 1/n\,(\mathrm{mod}\,\mathbb{Z})$. Since $0 \longrightarrow \mathbb{Z}m \longrightarrow M$ is exact and $\mathbb{Q}/\mathbb{Z}$ is injective, the map $f\colon \mathbb{Z}m \to \mathbb{Q}/\mathbb{Z}$ extends to a map $f\colon M \to \mathbb{Q}/\mathbb{Z}$, with $f(m) \neq 0$, as claimed. $\square$

Recall that if M is an R-module and N is any $\mathbb{Z}$-module, then $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ is an R^{op}-module under the R^{op}-action given by: For any $f \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$, and all $\gamma \in R$,

$$(f\gamma)(m) = f(\gamma m).$$

Similarly, if M is an R^{op}-module and N is any $\mathbb{Z}$-module, then $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ is an R-module under the R-action given by:

$$(\gamma f)(m) = f(m\gamma).$$

Then, $M^D = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is an R^{op}-module if M is an R-module (resp. an R-module if M is an R^{op}-module). Furthermore, the $\mathbb{Z}$-injection, $M \longrightarrow M^{DD}$, is an R-injection, The crux of Godement's proof is the following proposition:

Proposition 2.33 *If M is a projective R^{op}-module, then M^D is an injective R-module.*

Proof. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & X' \\ & & \Big\downarrow{\scriptstyle\varphi} & & \\ & & M^D & & \end{array}$$

where the row is exact. To prove that M^D is injective, we need to prove that φ extends to a map $\varphi' \colon X' \to M^D$. The map φ yields the map $M^{DD} \longrightarrow X^D$, and since we have an injection $M \longrightarrow M^{DD}$, we get a map $\theta \colon M \to X^D$. Now, since $\mathbb{Q}/\mathbb{Z}$ is injective, $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ maps the exact sequence

$$0 \longrightarrow X \longrightarrow X'$$

to the exact sequence $X'^D \longrightarrow X^D \longrightarrow 0$. So, we have the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \Big\downarrow{\scriptstyle\theta} & & \\ X'^D & \longrightarrow & X^D & \longrightarrow & 0, \end{array}$$

where the row is exact, and since M is projective, the map θ lifts to a map $\theta' \colon M \to X'^D$. Consequently, we get a map $X'^{DD} \longrightarrow M^D$, and since we have an injection $X' \longrightarrow X'^{DD}$, we get a map $X' \longrightarrow M^D$ extending φ, as desired. Therefore, M^D is injective. □

We can now prove Theorem 2.31. Consider the R^{op}-module M^D. We know that there is a free R^{op}-module, F, so that

$$F \longrightarrow M^D \longrightarrow 0 \quad \text{is exact.}$$

But, F being free is projective. We get the exact sequence

$$0 \longrightarrow M^{DD} \longrightarrow F^D.$$

By Proposition 2.33, the module F^D is injective. Composing the natural injection $M \longrightarrow M^{DD}$ with the injection $M^{DD} \longrightarrow F^D$, we obtain our injection, $M \longrightarrow F^D$, of M into an injective. □

Corollary 2.34 *Every R-module, M, has an injective resolution*

$$0 \longrightarrow M \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \cdots,$$

where the Q_i's are injective and the sequence is exact.

How about minimal injections? Recall that $N \longrightarrow M$ is a minimal (essential) injection iff N is large in M, which means that for any $L \subseteq M$, if $N \cap L = (0)$, then $L = (0)$.

We have the following characterization of essential injections, analogous to the characterization of minimal surjections:

Proposition 2.35 *The following are equivalent for injections $\theta \colon N \to M$:*

(1) $N \xrightarrow{\theta} M$ is essential.

(2) Given any module, Z, and any map, $M \xrightarrow{\varphi} Z$, if $N \longrightarrow M \xrightarrow{\varphi} Z$ is injective, then φ is injective.

(3) $\mathrm{Ker}\,(N \longrightarrow M \longrightarrow Z) = (0)$ *implies* $\mathrm{Ker}\,(M \longrightarrow Z) = (0)$*, for any module, Z.*

Proof. (DX)

In contradistinction to the case of covering surjections, essential injections always exist.

Proposition 2.36 *Given an injection, $N \longrightarrow M$, there exists a submodule, K, of M so that*

(1) The sequence $0 \longrightarrow N \longrightarrow M/K$ is exact, and

(2) It is an essential injection.

Proof. Let

$$\mathcal{S} = \{K \subseteq M \mid K \cap N = (0)\}.$$

Since $(0) \in \mathcal{S}$, the set $\mathcal{S}$ is nonempty. Partially order $\mathcal{S}$ by inclusion. If $\{Z_\alpha\}_\alpha$ is a chain in $\mathcal{S}$, let $Z = \bigcup_\alpha Z_\alpha$, a submodule of M. We have

$$Z \cap N = \left(\bigcup_\alpha Z_\alpha\right) \cap N = \bigcup_\alpha (Z_\alpha \cap N) = (0),$$

since $Z_\alpha \cap N) = (0)$, for all α. Therefore, $\mathcal{S}$ is inductive, and by Zorn's lemma, it has a maximal element, say K. Since $K \cap N = (0)$, property (1) is satisfied. For (2), take $L \subseteq M/K$ so that $L \cap \mathrm{Im}\,(N) = (0)$. We must show that $L = (0)$. By the second homomorphism theorem, L corresponds to $\widetilde{L}$ in M, with $K \subseteq \widetilde{L} \subseteq M$, and we are reduced to proving that $\widetilde{L} = K$.

Claim: For every $\eta \in \widetilde{L}$, if $\eta \notin K$, then $\eta \notin N$.

If $\eta \in \widetilde{L}$ and $\eta \notin K$ and $\eta \in N$, then $\overline{\eta} \in L \cap \mathrm{Im}\,(N)$, and so, $\overline{\eta} = 0$, since $L \cap \mathrm{Im}\,(N) = (0)$. (As usual, $\eta \mapsto \overline{\eta}$, denotes the canonical map $M \longrightarrow M/K$.) Yet $\eta \notin K$, a contradiction; the claim holds.

Assume that $\xi \in \widetilde{L}$ and $\xi \notin K$. Consider $K + R\xi$, a submodule of $\widetilde{L}$ strictly containing K. Since K is a maximal module with $K \cap N = (0)$, there is some $\eta \in (K + R\xi) \cap N$, with $\eta \neq 0$. Consequently, we have $\eta \in \widetilde{L}$ and $\eta \in N$. Now, if $\eta \in K$, then $\eta \in N \cap K = (0)$, contradicting the fact that $\eta \neq 0$; so, we must have $\eta \notin K$. However, this contradicts the claim. Therefore, ξ cannot exist, and $\widetilde{L} = K$. $\square$

Terminology: The module Q is an *injective hull of M* iff

(1) $M \longrightarrow Q$ is an essential injection, and

(2) The module Q is injective.

Theorem 2.37 *(Baer–Eckmann–Schopf) Every R-module has an injective hull.*

Proof. By Baer's embedding theorem (Theorem 2.31), there is an injective module, Q, so that $0 \longrightarrow M \longrightarrow Q$ is exact. Set

$$\mathcal{S} = \{L \mid M \subseteq L \subseteq Q \quad \text{and } 0 \longrightarrow M \longrightarrow L \text{ is essential}\}.$$

Since $M \in \mathcal{S}$, the set $\mathcal{S}$ is nonempty. The set $\mathcal{S}$ is partially ordered by inclusion, and it is inductive (DX). By Zorn's lemma, $\mathcal{S}$ has a maximal element, say L. I claim that L is injective. Look at the exact sequence $0 \longrightarrow L \longrightarrow Q$. By the argument in the previous proposition on essential extensions, there is a maximal $K \subseteq Q$, so that $K \cap L = (0)$ and $0 \longrightarrow L \longrightarrow Q/K$ is essential. Look at the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \longrightarrow & Q/K \\ & & \big\downarrow{\scriptstyle \varphi} & & \\ & & Q & & \end{array}.$$

Since Q is injective, there is a map, $\psi\colon Q/K \to Q$, extending φ; let $T = \mathrm{Im}\,\psi$. The map ψ is injective, because $\psi \restriction L$ is injective and the row is essential. Thus, $\psi\colon Q/K \to T$ is an isomorphism; moreover, $L \subseteq T$.

We contend that $T = L$. To see this, we will prove that $0 \longrightarrow M \longrightarrow T$ is essential. Now, being essential is a transitive property (DX); since T is essential over L (because $Q/K \cong T$ and Q/K is essential over L) and L is essential over M, we see that T is essential over M. But, L is maximal essential over M (in Q) and $L \subseteq T$; so, we conclude that $T = L$. Therefore, $L \cong Q/K$ and we have the maps

$$Q \longrightarrow Q/K \cong L \quad \text{and} \quad L \longrightarrow Q.$$

It follows that the sequence

$$0 \longrightarrow K \longrightarrow Q \longrightarrow L \longrightarrow 0$$

splits. Consequently, L is also injective; so, L is the required injective hull. □

Proposition 2.38 *(Uniqueness of projective covers and injective hulls.) Say $P \longrightarrow M$ is a projective cover and $\widetilde{P} \longrightarrow M$ is another surjection with $\widetilde{P}$ projective. Then, there exist $\widetilde{P}', \widetilde{P}'' \subseteq \widetilde{P}$, both projective so that*

(a) $\widetilde{P} = \widetilde{P}' \amalg \widetilde{P}''$.

(b) $P \cong \widetilde{P}'$.

(c) In the diagram

$$\begin{array}{ccccc} & & \widetilde{P} & & \\ & & \downarrow{\scriptstyle \widetilde{p}} & & \\ P & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

there are maps $\pi \colon \widetilde{P} \to P$ and $i \colon P \to \widetilde{P}$ in which π is surjective and i is injective, $\widetilde{P}'' = \mathrm{Ker}\, \pi$, $\widetilde{P}' = \mathrm{Im}\, i$ and $\widetilde{p} \restriction \widetilde{P}' \colon \widetilde{P}' \to M$ is a projective cover.

If M and $\widetilde{M}$ are isomorphic modules, then every isomorphism, $\theta \colon M \to \widetilde{M}$, extends to an isomorphism of projective covers, $P \longrightarrow \widetilde{P}$. The same statements hold for injective hulls and injections, $M \longrightarrow \widetilde{Q}$, where $\widetilde{Q}$ is injective, mutatis mutandis.

Proof. As $\widetilde{P}$ is projective, there is a map $\pi \colon \widetilde{P} \to P$, making the diagram commute. We claim that the map π is surjective. To see this, observe that $p(\mathrm{Im}\, \pi) = \mathrm{Im}\, \widetilde{p} = M$. Hence, $\mathrm{Im}\, \pi = P$, as P is a covering surjection. As P is projective and π is a surjection, π splits, i.e., there is a map $i \colon P \to \widetilde{P}$ and $\pi \circ i = \mathrm{id}_P$; it easily follows that i is injective. Define $\widetilde{P}'' = \mathrm{Ker}\, \pi$ and $\widetilde{P}' = \mathrm{Im}\, i$. We know that $i \colon P \to \widetilde{P}'$ is an isomorphism, and

$$0 \longrightarrow \mathrm{Ker}\, \pi\, (= \widetilde{P}'') \longrightarrow \widetilde{P} \longrightarrow P\, (\cong \widetilde{P}') \longrightarrow 0 \quad \text{is split exact;}$$

so, we deduce that $\widetilde{P} = \widetilde{P}' \amalg \widetilde{P}''$. The rest is clear.

For injectives, turn the arrows around, replace coproducts by products, etc. (DX). □

2.5 The Five Lemma and the Snake Lemma

Proposition 2.39 *(The five lemma.) Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} & & \downarrow{\scriptstyle \varphi_5} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5, \end{array}$$

then

(a) If φ_2 and φ_4 are injective and φ_1 is surjective, then φ_3 is injective.

(b) If φ_2 and φ_4 are surjective and φ_5 is injective, then φ_3 is surjective.

(c) If φ_1, φ_2, φ_4 , φ_5 are isomorphisms, then so is φ_3.

Proof. Obviously, (a) and (b) imply (c). Both (a) and (b) are proved by chasing the diagram (DX). □

Proposition 2.40 *(The snake lemma.) Given a commutative diagram with exact rows*

$$\begin{array}{ccccccc} & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & & \end{array},$$

then there exists a six term exact sequence

$$\text{Ker } \delta_1 \longrightarrow \text{Ker } \delta_2 \longrightarrow \text{Ker } \delta_3 \stackrel{\delta}{\longrightarrow} \text{Coker } \delta_1 \longrightarrow \text{Coker } \delta_2 \longrightarrow \text{Coker } \delta_3,$$

(where δ is called the connecting homomorphism) and if $M_1 \longrightarrow M_2$ is injective, so is Ker $\delta_1 \longrightarrow$ Ker δ_2*, while if $N_2 \longrightarrow N_3$ is surjective, so is* Coker $\delta_2 \longrightarrow$ Coker δ_3*.*

Proof. Simple diagram chasing shows Ker $\delta_1 \longrightarrow$ Ker $\delta_2 \longrightarrow$ Ker δ_3 is exact and Coker $\delta_1 \longrightarrow$ Coker $\delta_2 \longrightarrow$ Coker δ_3 is also exact (DX). Moreover, it also shows the very last assertions of the proposition.

We have to construct the connecting homomorphism, δ. Consider the commutative diagram:

$$\begin{array}{ccccccccc} & & \text{Ker } \delta_1 & \longrightarrow & \text{Ker } \delta_2 & \longrightarrow & \text{Ker } \delta_3 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M_1 & \longrightarrow & M_2 & \stackrel{p}{\longrightarrow} & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} & & \\ 0 & \longrightarrow & N_1 & \stackrel{i}{\longrightarrow} & N_2 & \longrightarrow & N_3 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \text{Coker } \delta_1 & \longrightarrow & \text{Coker } \delta_2 & \longrightarrow & \text{Coker } \delta_3 & & \end{array}.$$

Pick $\xi \in$ Ker δ_3, and consider ξ as an element of M_3. There is some $\eta \in M_2$ so that $p(\eta) = \xi$. So, we have $\delta_2(\eta) \in N_2$, and Im $\delta_2(\eta)$ in N_3 is $\delta_3(\xi) = 0$. As the lower row is exact and i is injective, η gives a unique $x \in N_1$, with $i(x) = \delta_2(\eta)$. We define our $\delta(\xi)$ as the projection of x on Coker δ_1. However, we need to check that this map is well-defined.

If we chose a different element, say $\widetilde{\eta}$, from η, where $p(\eta) = p(\widetilde{\eta}) = \xi$, then the construction is canonical from there on. Take $\delta_2(\eta)$ and $\delta_2(\widetilde{\eta})$. Since $\eta - \widetilde{\eta}$ goes to zero under p, there is some $y \in M_1$, so that $\eta - \widetilde{\eta} = \text{Im}\,(y)$ in M_2. Consequently $\eta = \widetilde{\eta} + \text{Im}\,(y)$; so, $\delta_2(\eta) = \delta_2(\widetilde{\eta}) + \delta_2(\text{Im}\,(y))$. But, $\delta_2(\text{Im}\,(y)) = i(\delta_1(y))$, and so,

$$\delta_2(\eta) = \delta_2(\widetilde{\eta}) + i(\delta_1(y)). \qquad (*)$$

As before, we have some unique elements x and $\widetilde{x}$ in N_1, so that $i(x) = \delta_2(\eta)$ and $i(\widetilde{x}) = \delta_2(\widetilde{\eta})$; so, by $(*)$, we get $i(x) = i(\widetilde{x}) + i(\delta_1(y))$. As i is injective, we conclude that

$$x = \widetilde{x} + \delta_1(y);$$

so, x and $\widetilde{x}$ have equal projections in Coker δ_1, and our definition of $\delta(\xi)$ is independent of the lift, η, of ξ to M_2. The rest is tedious diagram chasing (DX). □

Remark: As we said in Section 2.3, Proposition 2.17 also holds under slightly more general assumptions and its proof is a very nice illustration of the snake lemma. Here it is:

Proposition 2.41 *Let*

$$0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

be an exact sequence of Λ*-modules. If* M *is f.g. and* M'' *is f.p., then,* M' *is f.g.*

Proof. Let

$$F_1 \longrightarrow F_0 \longrightarrow M'' \longrightarrow 0$$

be a finite presentation of M'' (so, F_0, F_1 are free and f.g.) Consider the diagram

$$\begin{array}{ccccccccc} & & F_1 & \longrightarrow & F_0 & \longrightarrow & M'' & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0. \end{array}$$

Now, F_0 is free, so there exists a map $F_0 \longrightarrow M$ lifting the surjection $F_0 \longrightarrow M''$. Call this map θ. From the commutative diagram which results when θ is added, we deduce a map $\gamma\colon F_1 \to M'$. Hence, we find the bigger commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & F_1 & \longrightarrow & F_0 & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \longrightarrow & M' & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \text{Coker } \gamma & \longrightarrow & \text{Coker } \theta & \longrightarrow & 0 & & \end{array}$$

But, by the snake lemma, Coker $\gamma \cong$ Coker θ. However, Coker θ is f.g. as M is f.g. The image of γ is f.g. as F_1 is f.g. And now, M' is caught between the f.g. modules Im γ and Coker γ; so, M' is f.g. □

2.6 Tensor Products and Flat Modules

Let R be a ring (not necessarily commutative). In this section, to simplify the notation, the product of R-modules, M and N, *viewed as sets*, will be denoted $M \times N$, instead of $M \prod_{\mathcal{S}\mathrm{ets}} N$. For any R^{op}-module, M, any R-module, N, and any abelian group, Z, we set

$$\mathrm{Bi}_R(M,N;Z) = \left\{ \varphi\colon M \times N \longrightarrow Z \;\middle|\; \begin{array}{l} (1)\ (\forall m, m' \in M)(\forall n \in N)(\varphi(m+m',n) = \varphi(m,n) + \varphi(m',n)) \\ (2)\ (\forall m \in M)(\forall n, n' \in N)(\varphi(m, n+n') = \varphi(m,n) + \varphi(m,n')) \\ (3)\ (\forall m \in M)(\forall n \in N)(\forall r \in R)(\varphi(mr,n) = \varphi(m,rn)) \end{array} \right\}.$$

Observe that

(1) The set $\mathrm{Bi}_R(M,N;Z)$ is an abelian group under addition; i.e., if $\varphi, \psi \in \mathrm{Bi}_R(M,N;Z)$, then $\varphi + \psi \in \mathrm{Bi}_R(M,N;Z)$.

(2) The map $Z \rightsquigarrow \mathrm{Bi}_R(M,N;Z)$ is a functor from $\mathcal{A}\mathrm{b}$ to $\mathcal{S}\mathrm{ets}$. Is this functor representable? To be more explicit, does there exist an abelian group, $T(M,N)$, and an element, $\Phi \in \mathrm{Bi}_R(M,N;T(M,N))$, so that the pair $(T(M,N),\Phi)$ represents $\mathrm{Bi}_R(M,N;-)$, i.e., the map

$$\mathrm{Hom}_{\mathbb{Z}}(T(M,N),Z) \xrightarrow{\sim} \mathrm{Bi}_R(M,N;Z)$$

via $\varphi \mapsto \varphi \circ \Phi$, is a functorial isomorphism?

Theorem 2.42 *The functor* $Z \rightsquigarrow \mathrm{Bi}_R(M,N;Z)$ *from* $\mathcal{A}\mathrm{b}$ *to* $\mathcal{S}\mathrm{ets}$ *is representable.*

Proof. Write $\mathcal{F}$ for the free abelian group on the set $M \times N$. Recall that $\mathcal{F}$ consists of formal sums

$$\sum_\alpha \xi_\alpha (m_\alpha, n_\alpha),$$

where $\xi_\alpha \in \mathbb{Z}$, with $\xi_\alpha = 0$ for all but finitely many α's, and with $m_\alpha \in M$ and $n_\alpha \in N$. Consider the subgroup, $\mathcal{N}$, of $\mathcal{F}$ generated by the elements

$$\begin{aligned} &(m_1+m_2, n) - (m_1, n) - (m_2, n) \\ &(m_1, n_1+n_2) - (m, n_1) - (m, n_2) \\ &(mr, n) - (m, rn). \end{aligned}$$

Form $\mathcal{F}/\mathcal{N}$ and write $m \otimes_R n$ for the image of (m,n) in $\mathcal{F}/\mathcal{N}$. We have

(α) $(m_1+m_2) \otimes_R n = m_1 \otimes_R n + m_2 \otimes_R n$.

(β) $m \otimes_R (n_1+n_2) = m \otimes_R n_1 + m \otimes_R n_2$.

(γ) $(mr) \otimes_R n = m \otimes_R (rn)$.

Let $T(M,N) = \mathcal{F}/\mathcal{N}$ and let Φ be given by $\Phi(m,n) = m \otimes_R n$. Then, ($\alpha$), ($\beta$), ($\gamma$) imply that Φ belongs to $\mathrm{Bi}_R(M,N;T(M,N))$, and the assignment, $\varphi \mapsto \varphi \circ \Phi$, gives the functorial map

$$\mathrm{Hom}_{\mathbb{Z}}(T(M,N),Z) \longrightarrow \mathrm{Bi}_R(M,N;Z).$$

We need to prove that this map is an isomorphism. Pick $\theta \in \mathrm{Bi}_R(M,N;Z)$; we claim that θ yields a homomorphism, $T(M,N) \longrightarrow Z$. Such a homomorphism is merely a homomorphism, $\mathcal{F} \longrightarrow Z$, that vanishes on $\mathcal{N}$. But, $\mathcal{F}$ is free; so we just need to know the images of the basis elements, (m,n), in Z. For this, map (m,n) to $\theta(m,n)$. The induced homomorphism vanishes on the generators of $\mathcal{N}$, as θ is bilinear; thus, θ yields a map

$$\Xi(\theta)\colon \mathcal{F}/\mathcal{N} \longrightarrow Z,$$

and we get our inverse map $\mathrm{Bi}_R(M,N;Z) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(T(M,N),Z)$. Routine checking shows that the maps $\varphi \mapsto \varphi \circ \Phi$ and $\theta \mapsto \Xi(\theta)$ are functorial and mutual inverses. $\square$

Definition 2.7 The group, $T(M,N) = \mathcal{F}/\mathcal{N}$, constructed in Theorem 2.42, is called the *tensor product of M and N over R* and is denoted $M \otimes_R N$.

Remark: Note that Theorem 2.42 says two things:

(1) For every $\mathbb{Z}$-linear map, $f\colon M \otimes_R N \to Z$, the map, φ, given by $\varphi(m,n) = f(m \otimes n)$, for all $m \in M$ and $n \in N$, is bilinear (i.e., $\varphi \in \mathrm{Bi}_R(M,N;Z)$), and

(2) For every bilinear map, $\varphi \in \mathrm{Bi}_R(M,N;Z)$, there is a *unique* $\mathbb{Z}$-linear map, $f\colon M \otimes_R N \to Z$, with $\varphi(m,n) = f(m \otimes n)$, for all $m \in M$ and $n \in N$. In most situations, this is the property to use in order to define a map from a tensor product to another module.

One should avoid "looking inside" a tensor product, especially when defining maps. Indeed, given some element $w \in M \otimes_R N$, there may be different pairs, $(m,n) \in M \times N$ and $(m',n') \in M \times N$, with $w = m \otimes_R n = m' \otimes_R n'$. Worse, one can have $m \otimes_R n = \sum_\alpha m_\alpha \otimes_R n_\alpha$. Thus, defining a function as $f(m \otimes_R n)$ for all $m \in M$ and $n \in N$ usually does not make sense; there is no guarantee that $f(m \otimes_R n)$ and $f(m' \otimes_R n')$ should agree when $m \otimes_R n = m' \otimes_R n'$. The "right way" to define a function on $M \otimes_R N$ is to first define a function, φ, on $M \times N$, and then to check that φ is bilinear (i.e., $\varphi \in \mathrm{Bi}_R(M,N;Z)$). Then, there is a unique homomorphism, $f\colon M \otimes_R N \to Z$, so that $f(m \otimes_R n) = \varphi(m,n)$. Having shown that f exists, we now may safely use its description in terms of elements, $m \otimes n$, since they generate $M \otimes_R N$. We will have many occasions to use this procedure in what follows.

Basic properties of the tensor product:

Proposition 2.43 *The tensor product, $M \otimes_R N$, is a functor of each variable (from R^{op}-modules to $\mathcal{A}\mathrm{b}$ or from R-modules to $\mathcal{A}\mathrm{b}$). Moreover, as a functor, it is right-exact.*

Proof. Just argue for M, the argument for N being similar. Say $f\colon M \to \widetilde{M}$ is an R^{op}-morphism. Consider $M \times N$ and the map: $\widetilde{f}(m,n) = f(m) \otimes n$. This is clearly a bilinear map $M \times N \longrightarrow \widetilde{M} \otimes_R N$. By the defining property of $M \otimes_R N$, we obtain our map (in $\mathcal{A}\mathrm{b}$) $M \otimes_R N \longrightarrow \widetilde{M} \otimes_R N$. Consequently, **now that we know the map is defined**, we see that it is given by

$$m \otimes n \mapsto f(m) \otimes n.$$

For right-exactness, again vary M (the proof for N being similar). Consider the exact sequence

$$M' \xrightarrow{\ i\ } M \longrightarrow M'' \longrightarrow 0. \qquad (\dagger)$$

We must prove that

$$M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0 \quad \text{is exact.} \qquad (\dagger\dagger)$$

Pick a test abelian group, Z, and write C for $\mathrm{Coker}\,(M' \otimes_R N \longrightarrow M \otimes_R N)$. We have the exact sequence

$$M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow C \longrightarrow 0. \qquad (*)$$

Now, $\mathrm{Hom}_{\mathcal{A}\mathrm{b}}(-,Z)$ is left-exact, so we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}\mathrm{b}}(C,Z) \longrightarrow \mathrm{Hom}_{\mathcal{A}\mathrm{b}}(M \otimes_R N, Z) \xrightarrow{\ i^*\ } \mathrm{Hom}_{\mathcal{A}\mathrm{b}}(M' \otimes_R N, Z). \qquad (**)$$

The two terms on the righthand side are isomorphic to $\mathrm{Bi}_R(M,N;Z)$ and $\mathrm{Bi}_R(M',N;Z)$, and the map, i^*, is

$$\varphi \in \mathrm{Bi}_R(M,N;Z) \mapsto i^*\varphi \in \mathrm{Bi}_R(M',N;Z), \quad \text{where } i^*\varphi(m',n) = \varphi(i(m'),n).$$

When is $i^*\varphi = 0$? Observe that $i^*\varphi = 0$ iff $\varphi(i(m'), n) = 0$ for all $m' \in M'$ and all $n \in N$. So, $\mathrm{Hom}_{\mathcal{A}\mathrm{b}}(C, Z)$ is the subgroup of $\mathrm{Bi}_R(M, N; Z)$ given by

$$\{\varphi \in \mathrm{Bi}_R(M, N; Z) \mid (\forall m' \in M')(\forall n \in N)(\varphi(i(m'), n) = 0)\},$$

and denoted $\mathrm{Bi}^*_R(M, N; Z)$.

Claim: There is a canonical (functorial in Z) isomorphism

$$\mathrm{Bi}^*_R(M, N; Z) \cong \mathrm{Bi}_R(M'', N; Z).$$

Say $\varphi \in \mathrm{Bi}^*_R(M, N; Z)$. Pick $\overline{m} \in M''$ and $n \in N$, choose any $m \in M$ lifting $\overline{m}$ and set

$$\psi(\overline{m}, n) = \varphi(m, n).$$

If $\widetilde{m}$ is another lift, then, as (†) is exact, $\widetilde{m} - m = i(m')$ for some $m' \in M'$. So, $\varphi(\widetilde{m} - m, n) = 0$, as $\varphi \in \mathrm{Bi}^*_R(M, N; Z)$. But, $\varphi(\widetilde{m} - m, n) = \varphi(\widetilde{m}, n) - \varphi(m, n)$, and so, $\varphi(\widetilde{m}, n) = \varphi(m, n)$, which proves that ψ is well-defined. Consequently, we have the map $\varphi \mapsto \psi$ from $\mathrm{Bi}^*_R(M, N; Z)$ to $\mathrm{Bi}_R(M'', N; Z)$. If $\psi \in \mathrm{Bi}_R(M'', N; Z)$, pick any $m \in M$ and $n \in N$ and set $\varphi(m, n) = \psi(\overline{m}, n)$ (where $\overline{m}$ is the image of m in M''). These are inverse maps. Therefore, we obtain the isomorphism

$$\mathrm{Bi}^*_R(M, N; Z) \cong \mathrm{Bi}_R(M'', N; Z),$$

functorial in Z, as claimed. However, the righthand side is isomorphic to $\mathrm{Hom}_{\mathcal{A}\mathrm{b}}(M'' \otimes_R N, Z)$, and so, by Yoneda's lemma, we see that $C \cong M'' \otimes_R N$, and (††) is exact. □

Proposition 2.44 *Consider R as R^{op}-module. Then, $R \otimes_R M \xrightarrow{\sim} M$. Similarly, if R is considered as R-module, then $M \otimes_R R \xrightarrow{\sim} M$. Say $M = \coprod_{i=1}^t M_i$, then*

$$M \otimes_R N \cong \coprod_{i=1}^t (M_i \otimes_R N).$$

(Similarly for N.)

Proof. We treat the first case $R \otimes_R M \xrightarrow{\sim} M$, the second one being analogous. Pick a test group, Z, and look at $\mathrm{Hom}_{\mathcal{A}\mathrm{b}}(R \otimes_R M, Z) \cong \mathrm{Bi}_R(R, M; Z)$. Any $\varphi \in \mathrm{Bi}_R(R, M; Z)$ satisfies $\varphi(r, m) = \varphi(1, rm)$, by bilinearity. Now, set $\varphi_0(m) = \varphi(1, m)$. Then, as φ is bilinear, we deduce that $\varphi_0 \colon M \to Z$ is a group homomorphism. The map $\varphi \mapsto \varphi_0$ is clearly an isomorphism from $\mathrm{Bi}_R(R, M; Z)$ to $\mathrm{Hom}_R(M, Z)$, functorial in Z, and so, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{A}\mathrm{b}}(R \otimes_R M, Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}\mathrm{b}}(M, Z)$$

functorial in Z. By Yoneda's lemma, we get the isomorphism $R \otimes_R M \xrightarrow{\sim} M$.

For coproducts, we use an induction on t. The base case, $t = 1$, is trivial. For the induction step, look at the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow \coprod_{j=2}^t M_j \longrightarrow 0.$$

This sequence is not only exact, but split exact. Now, from this, tensoring with N on the right and using the induction hypothesis, we get another split exact sequence (DX)

$$0 \longrightarrow M_1 \otimes_R N \longrightarrow M \otimes_R N \longrightarrow \coprod_{j=2}^t (M_j \otimes_R N) \longrightarrow 0;$$

so,

$$M \otimes_R N \cong \coprod_{i=1}^{t} (M_i \otimes_R N). \qquad \square$$

In the next section we will prove that tensor product commutes with arbitrary coproducts.

Computation of some tensor products:

(1) Say $F = \coprod_S R$, as R^{op}-module (with S finite). Then,

$$F \otimes_R N = (\coprod_S R) \otimes_R N \cong \coprod_S (R \otimes_R N) \cong \coprod_S N.$$

Similarly, $M \otimes_R F \cong \coprod_S M$, if $F = \coprod_S R$, as R-module (with S finite).

(1a) Assume G is also free, say $G = \coprod_T R$ (with T finite), as an R-module. Then,

$$F \otimes_R G \cong \coprod_S G = \coprod_S \coprod_T R = \coprod_{S \times T} R.$$

(2) Say $\mathfrak{A}$ is an R^{op}-ideal of R. Then $(R/\mathfrak{A}) \otimes_R M \cong M/\mathfrak{A}M$. Similarly, if $\mathfrak{A}$ is an R-ideal of R, then for any R^{op}-module, M, we have $M \otimes_R (R/\mathfrak{A}) \cong M/M\mathfrak{A}$. (These are basic results.)

Proof. We have the exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow R/\mathfrak{A} \longrightarrow 0,$$

where $\mathfrak{A}$ is an R^{op}-ideal. By tensoring on the right with M, we get the right-exact sequence

$$\mathfrak{A} \otimes_R M \longrightarrow R \otimes_R M \longrightarrow (R/\mathfrak{A}) \otimes_R M \longrightarrow 0.$$

Consider the diagram:

$$\begin{array}{ccccccccc}
 & & \mathfrak{A} \otimes_R M & \longrightarrow & R \otimes_R M & \longrightarrow & (R/\mathfrak{A}) \otimes_R M & \longrightarrow & 0 \\
 & & & & \big\downarrow & & & & \\
0 & \longrightarrow & \mathfrak{A}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{A}M & \longrightarrow & 0.
\end{array}$$

The middle vertical arrow is an isomorphism; we claim that there is a map $\mathfrak{A} \otimes_R M \longrightarrow \mathfrak{A}M$. Such a map corresponds to a bilinear map in $\mathrm{Bi}_R(\mathfrak{A}, M; \mathfrak{A}M)$. But, $(\alpha, m) \mapsto \alpha m$ is just such a bilinear map. So, we get our map $\mathfrak{A} \otimes_R M \longrightarrow \mathfrak{A}M$. Now, of course, it is given by $\alpha \otimes m \mapsto \alpha m$. But then, there is induced a righthand vertical arrow and we get the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \text{Ker } \rho & \longrightarrow & \text{Ker } w & \longrightarrow & \text{Ker } y & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathfrak{A} \otimes_R M & \longrightarrow & R \otimes_R M & \longrightarrow & (R/\mathfrak{A}) \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle y} & & \\
0 & \longrightarrow & \mathfrak{A}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{A}M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \text{Coker } \rho & \longrightarrow & \text{Coker } w & \longrightarrow & \text{Coker } y & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

The snake lemma yields an exact sequence

$$\text{Ker } w \longrightarrow \text{Ker } y \xrightarrow{\delta} \text{Coker } \rho \longrightarrow \text{Coker } w \longrightarrow \text{Coker } y \longrightarrow 0.$$

Since ρ is onto (DX), we have Coker $\rho = 0$, and since w is an isomorphism, we have Ker $w =$ Coker $w = 0$. Thus, Ker $y = 0$. As Coker $w \longrightarrow$ Coker $y \longrightarrow 0$ is exact and Coker $w = 0$, we deduce that Coker $y = 0$. Therefore, y is an isomorphism, as claimed. □ (One can also use the five lemma in the proof.)

(3) Compute $\mathbb{Z}/r\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/s\mathbb{Z}$.

We claim that the answer is $\mathbb{Z}/t\mathbb{Z}$, where $t = \text{g.c.d.}(r, s)$.

We know (DX) that $\otimes_R$ is an additive functor. From the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{r} \mathbb{Z} \longrightarrow \mathbb{Z}/r\mathbb{Z} \longrightarrow 0,$$

we get the exact sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/s\mathbb{Z}) \xrightarrow{r} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/s\mathbb{Z}) \longrightarrow (\mathbb{Z}/r\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/s\mathbb{Z}) \longrightarrow 0.$$

Write X for $(\mathbb{Z}/r\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/s\mathbb{Z})$. Hence,

$$\mathbb{Z}/s\mathbb{Z} \xrightarrow{r} \mathbb{Z}/s\mathbb{Z} \longrightarrow X \longrightarrow 0 \quad \text{is exact.}$$

Pick $\overline{z} \in \mathbb{Z}/s\mathbb{Z}$, and say $r\overline{z} = 0$, i.e., $rz \equiv 0 \pmod{s}$. We have $r = \rho t$ and $s = \sigma t$, with $\text{g.c.d.}(\rho, \sigma) = 1$. Now, $rz \equiv 0 \pmod{s}$ means that $rz = sk$, for some k; so, we have $\rho t z = \sigma t k$, for some k, and so, $\rho z = \sigma k$, for some k. We see that $\sigma \mid \rho z$, and since $\text{g.c.d.}(\rho, \sigma) = 1$, we conclude that $\sigma \mid z$. As a consequence, $\sigma t \mid tz$; so, $s\ (= \sigma t) \mid tz$ and we conclude that $t\overline{z} = 0$ in $\mathbb{Z}/s\mathbb{Z}$. Conversely, if $t\overline{z} = 0$, we get $\rho t \overline{z} = 0$, i.e., $r\overline{z} = 0$ in $\mathbb{Z}/s\mathbb{Z}$. Therefore, we have shown that

$$\text{Ker}\,(\text{mult. by } r) = \text{Ker}\,(\text{mult. by } t) \text{ in } \mathbb{Z}/s\mathbb{Z};$$

consequently (as this holds for no further divisor of t)

$$\text{Im}\,(\text{mult. by } r) = \text{Im}\,(\text{mult. by } t) \text{ in } \mathbb{Z}/s\mathbb{Z}.$$

Thus,

$$X \cong (\mathbb{Z}/s\mathbb{Z})/(t\mathbb{Z}/s\mathbb{Z}) \cong \mathbb{Z}/t\mathbb{Z}.$$

(4) Say M is an S-module and an R^{op}-module. If

$$(sm)r = s(mr), \quad \text{for all } s \in S \text{ and all } r \in R,$$

then M is called an (S, R^{op})-*bimodule*, or simply a *bimodule* when reference to S and R are clear. We will always assume that if M is an S-module and an R^{op}-module, then it is a bimodule.

If M is a (S, R^{op})-bimodule and N is an R-module, we claim that $M \otimes_R N$ has a natural structure of S-module.

Illegal procedure: $s(m \otimes_R n) = (sm) \otimes_R n$.

The correct way to proceed is to pick any $s \in S$ and to consider the map, φ_s, from $M \times N$ to $M \otimes_R N$ defined by

$$\varphi_s(m, n) = (sm) \otimes n.$$

It is obvious that this map is bilinear (in m and n).

Remark: (The reader should realize that the bimodule structure of M is used here to check property (3) of bilinearity. We have

$$\varphi_s(mr, n) = (s(mr)) \otimes n = ((sm)r \otimes n = (sm) \otimes rn = \varphi_s(m, rn).)$$

So, we get a map $M \otimes_R N \longrightarrow M \otimes_R N$, corresponding to s. Check that this gives the (left) action of S on $M \otimes_R N$. Of course, it is

$$s(m \otimes_R n) = (sm) \otimes_R n.$$

Similarly, if M is an R^{op}-module and N is a (R, S^{op})-bimodule, then $M \otimes_R N$ is an S^{op}-module; the (right) action of S is

$$(m \otimes_R n)s = m \otimes_R (ns).$$

Remark: If M is an R-module, N is an (R, S^{op})-bimodule, and Z is an S^{op}-module, then any S^{op}-linear map $f\colon M \otimes_R N \longrightarrow Z$ satisfies the property:

$$f(m \otimes_R (ns)) = f(m \otimes_R n)s, \quad \text{for all } s \in S,$$

since $f(m \otimes_R (ns)) = f((m \otimes_R n)s) = f(m \otimes_R n)s$. Thus, the corresponding bilinear map $\varphi\colon M \times N \longrightarrow Z$ defined by

$$\varphi(m, n) = f(m \otimes_R n)$$

satisfies the property:

$$\varphi(m, ns) = \varphi(m, n)s, \quad \text{for all } s \in S.$$

This suggests defining a set, $S^{\mathrm{op}}\text{-Bi}_R(M, N; Z)$, by

$$S^{\mathrm{op}}\text{-Bi}_R(M, N; Z) = \left\{ \varphi\colon M \times N \longrightarrow Z \;\middle|\; \begin{array}{l} (1)\ (\forall m, m' \in M)(\forall n \in N) \\ \quad (\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)) \\ (2)\ (\forall m \in M)(\forall n, n' \in N) \\ \quad (\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')) \\ (3)\ (\forall m \in M)(\forall n \in N)(\forall r \in R)(\varphi(mr, n) = \varphi(m, rn)) \\ (4)\ (\forall m \in M)(\forall n \in N)(\forall s \in S)(\varphi(m, ns) = \varphi(m, n)s) \end{array} \right\}.$$

Then, we have

Theorem 2.45 *Let M be an R-module and N be an (R, S^{op})-bimodule. The functor $Z \rightsquigarrow S^{\mathrm{op}}\text{-Bi}_R(M, N; Z)$ from $\mathcal{M}\mathrm{od}(S^{\mathrm{op}})$ to $\mathcal{S}\mathrm{ets}$ is representable by $(M \otimes_R N, \Phi)$, where Φ is given by $\Phi(m, n) = m \otimes_R n$.*

Note that the above statement includes the fact that $M \otimes_R N$ is an S^{op}-module.

Similarly, if M is an (S, R^{op})-bimodule, N is an R-module and Z is an S-module, then we can define the set, $S\text{-Bi}_R(M, N; Z)$, in an analogous way (replace (4) by $\varphi(sm, n) = s\varphi(m, n)$), and we find

Theorem 2.46 *Let M be an (S, R^{op})-bimodule and N be an R-module. The functor $Z \rightsquigarrow S\text{-Bi}_R(M, N; Z)$ from $\mathcal{M}\mathrm{od}(S)$ to $\mathcal{S}\mathrm{ets}$ is representable by $(M \otimes_R N, \Phi)$, where Φ is given by $\Phi(m, n) = m \otimes_R n$.*

Associativity of tensor: Let M be an R^{op}-module, N an (R, S^{op})-bimodule, and Z an S-module. Then,

$$(M \otimes_R N) \otimes_S Z \cong M \otimes_R (N \otimes_S Z).$$

For any test group, T, the left hand side represents the functor

$$T \rightsquigarrow \mathrm{Bi}_S(M \otimes_R N, Z; T)$$

and the righthand side represents the functor

$$T \rightsquigarrow \mathrm{Bi}_R(M, N \otimes_S Z; T).$$

We easily check that both these are just the trilinear maps, "$\mathrm{Tri}_{R,S}(M, N, Z; T)$;" so, by the uniqueness of objects representing functors, we get our isomorphism. In particular,

(A) $(M \otimes_R S) \otimes_S Z \cong M \otimes_R (S \otimes_S Z) \cong M \otimes_R Z$.

(B) Say $S \longrightarrow R$ is a given *surjective* ring map and say M is an R^{op}-module and N is an R-module. Then, M is an S^{op}-module, N is an S-module and

$$M \otimes_S N \cong M \otimes_R N.$$

To see this, look at $\mathcal{F}/\mathcal{N}$ and see that the same elements are identified.

(C) Say $S \longrightarrow R$ is a ring map. Then, $M \otimes_R N$ is a homomorphic image of $M \otimes_S N$.

Remark: *Adjointness Properties of tensor*: We observed that when M is an (S, R^{op})-bimodule and N is an R-module, then $M \otimes_R N$ is an S-module (resp. when M is an R^{op}-module and N is an (R, S^{op})-bimodule, then $M \otimes_R N$ is an S^{op}-module.) The abelian group $\mathrm{Hom}(M, N)$ also acquires various module structures depending on the bimodule structures of M and N. There are four possible module structures:

(a) The module M is an (R, S^{op})-bimodule and N is an R-module. Define an S-action on $\mathrm{Hom}_R(M, N)$ as follows: For every $f \in \mathrm{Hom}_R(M, N)$ and every $s \in S$,

$$(sf)(m) = f(ms), \quad \text{for all } m \in M.$$

(b) The module M is an (R, S^{op})-bimodule and N is an S^{op}-module. Define an R^{op}-action on $\mathrm{Hom}_{S^{\mathrm{op}}}(M, N)$ as follows: For every $f \in \mathrm{Hom}_{S^{\mathrm{op}}}(M, N)$ and every $r \in R$,

$$(fr)(m) = f(rm), \quad \text{for all } m \in M.$$

(c) The module M is an R^{op}-module and N is an (S, R^{op})-bimodule. Define an S-action on $\mathrm{Hom}_{R^{\mathrm{op}}}(M, N)$ as follows: For every $f \in \mathrm{Hom}_{R^{\mathrm{op}}}(M, N)$ and every $s \in S$,

$$(sf)(m) = s(f(m)), \quad \text{for all } m \in M.$$

(d) The module M is an S-module and N is an (S, R^{op})-bimodule. Define an R^{op}-action on $\mathrm{Hom}_S(M, N)$ as follows: For every $f \in \mathrm{Hom}_S(M, N)$ and every $r \in R$,

$$(fr)(m) = (f(m))r, \quad \text{for all } m \in M.$$

The reader should check that the actions defined in (a), (b), (c), (d) actually give corresponding module structures. Note how the contravariance in the left argument, M, of $\mathrm{Hom}(M, N)$ flips a left action into a right action, and conversely. As an example, let us check (a). For all $r, t \in S$,

$$((st)f)(m) = f(m(st)) = f((ms)t) = (tf)(ms) = (s(tf))(m).$$

We also need to check that sf is R-linear. This is where we use the bimodule structure of M. We have

$$(sf)(rm) = f((rm)s) = f(r(ms)) = rf(ms) = r((sf)(m)).$$

We are now ready to state an important adjointness relationship between Hom and $\otimes$.

Proposition 2.47 *If M is an R^{op}-module, N is an (R, S^{op})-bimodule, and Z is an S^{op}-module, then there is a natural functorial isomorphism*

$$\mathrm{Hom}_{S^{\mathrm{op}}}(M \otimes_R N, Z) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(M, \mathrm{Hom}_{S^{\mathrm{op}}}(N, Z)).$$

When M is an R-module, N is an (S, R^{op})-bimodule, and Z is an S-module, then there is a natural functorial isomorphism

$$\mathrm{Hom}_S(N \otimes_R M, Z) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, Z)).$$

Proof. Using Theorem 2.45, it is enough to prove that

$$S^{\mathrm{op}}\text{-}\mathrm{Bi}_R(M, N; Z) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(M, \mathrm{Hom}_{S^{\mathrm{op}}}(N, Z))$$

and using Theorem 2.46, to prove that

$$S\text{-}\mathrm{Bi}_R(N, M; Z) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, Z)).$$

We leave this as a (DX). □

Proposition 2.47 states that the functor $- \otimes_R N$ is left adjoint to the functor $\mathrm{Hom}_{S^{\mathrm{op}}}(N, -)$ when N is an (R, S^{op})-bimodule (resp. $N \otimes_R -$ is left adjoint to $\mathrm{Hom}_S(N, -)$ when N is an (S, R^{op})-bimodule).

Commutativity of tensor: If R is commutative, then $M \otimes_R N \cong N \otimes_R M$. The easy proof is just to consider $(m, n) \mapsto n \otimes m$. It is bilinear; so, we get a map $M \otimes_R N \longrightarrow N \otimes_R M$. Interchange M and N, then check the maps are mutually inverse.

(5) Let G be a torsion abelian group and Q a divisible abelian group. Then,

$$Q \otimes_{\mathbb{Z}} G = (0).$$

Look at $\mathrm{Hom}_{\mathbb{Z}}(Q \otimes_{\mathbb{Z}} G, T) \cong \mathrm{Bi}_{\mathbb{Z}}(Q, G; T)$, for any test group, T. Take $\varphi \in \mathrm{Bi}_{\mathbb{Z}}(Q, G; T)$ and look at $\varphi(q, \sigma)$. Since G is torsion, there is some n so that $n\sigma = 0$. But, Q is divisible, so $q = n\widetilde{q}$, for some $\widetilde{q} \in Q$. Thus,

$$\varphi(q, \sigma) = \varphi(n\widetilde{q}, \sigma) = \varphi(\widetilde{q}n, \sigma) = \varphi(\widetilde{q}, n\sigma) = 0.$$

As this holds for all q and σ, we have $\varphi \equiv 0$, and so, $Q \otimes_{\mathbb{Z}} G = (0)$.

(6) Free modules (again). Let $F = \coprod_S R$, an R^{op}-module and $G = \coprod_T R$, an R-module (with both S and T finite). We know that

$$F \otimes_R G = \coprod_{S \times T} R.$$

We want to look at this tensor product more closely. Pick a basis, $e_1, \ldots, e_s$, in F and a basis, $f_1, \ldots, f_t$, in G, so that

$$F = \coprod_{j=1}^{s} e_j R \quad \text{and} \quad G = \coprod_{l=1}^{t} R f_l.$$

Then, we get

$$F \otimes_R G = \coprod_{j=1,l=1}^{s,t} (e_j R) \otimes_R (R f_l).$$

Thus, we get copies of R indexed by elements $e_j \otimes f_l$. Suppose that F is also an R-module. This means that $\rho e_j \in F$ makes sense. We assume $\rho e_j \in e_j R$, that is the left action of R commutes with the coproduct decomposition. Then $F \otimes_R G$ is an R-module and it is free of rank st if the left action, ρe_j, has obvious properties (and similarly if G is also an R^{op}-module).

It is not true in general that $\rho e_j = e_j \rho$. Call a free module a *good free module* iff it possesses a basis $e_1, \ldots, e_s$ so that $\rho e_j = e_j \rho$, for all $\rho \in R$. (This is not standard terminology.)

It is not generally true even here, that

$$\rho m = m\rho \quad (m \in F).$$

Say $m = \sum_{j=1}^{s} e_j \lambda_j$. Then, we have

$$\rho m = \sum_{j=1}^{s} \rho(e_j \lambda_j) = \sum_{j=1}^{s} \rho(\lambda_j e_j) = \sum_{j=1}^{s} (\rho \lambda_j) e_j,$$

and

$$m\rho = \sum_{j=1}^{s} (e_j \lambda_j)\rho = \sum_{j=1}^{s} e_j (\lambda_j \rho) = \sum_{j=1}^{s} (\lambda_j \rho) e_j.$$

In general, $\rho \lambda_j \neq \lambda_j \rho$, and so, $\rho m \neq m \rho$.

Consider the special example in which $R = k =$ a field. Then, all modules are free and good. Let V be a k-vector space of dimension d, and let $e_1, \ldots, e_d$ be some basis for V. We know that the dual space, V^D, has the dual basis, $f_1, \ldots, f_d$, characterized by

$$f_i(e_j) = \delta_{ij}.$$

Every $v \in V$ can be uniquely written as $v = \sum \lambda_i e_i$, and every $f \in V^D$ can be uniquely written as $f = \sum \mu_i f_i$. Consider the space

$$\underbrace{V \otimes_k \cdots \otimes_k V}_{a} \otimes_k \underbrace{V^D \otimes_k \cdots \otimes_k V^D}_{b}.$$

Elements of this space, called (a, b)-*tensors*, have the unique form

$$\sum_{j_1, \ldots, j_b}^{i_1, \ldots, i_a} c_{j_1, \ldots, j_b}^{i_1, \ldots, i_a} e_{i_1} \otimes_k \cdots \otimes_k e_{i_a} \otimes_k f_{j_1} \otimes_k \cdots \otimes_k f_{j_b}.$$

So, $V \otimes_k \cdots \otimes_k V \otimes_k V^D \otimes_k \cdots \otimes_k V^D$ may be identified with tuples $(c_{j_1, \ldots, j_b}^{i_1, \ldots, i_a})$, of elements of k, doubly-multiply indexed. They transform as ... (change of basis). A tensor in $V \otimes_k \cdots \otimes_k V \otimes_k V^D \otimes_k \cdots \otimes_k V^D$ is *cogredient of rank (or degree) a and contragredient of rank (or degree) b*. A *tensor field* on a space, X, is a function (of some class, C^∞, C^k, holomorphic, etc.) from X to a tensor vector space, as above. More generally, it is a section of a tensor bundle over X. Also, we can apply f_{j_m} to e_{i_k} and reduce the cogredient and contragredient ranks by one each. This gives a map $V^{\otimes a} \otimes_R V^{D^{\otimes b}} \longrightarrow V^{\otimes (a-1)} \otimes_R V^{D^{\otimes (b-1)}}$, called *contraction*.

Remark: Let M be an R-module, N be an S-module, and Z be an (R, S^{op})-bimodule. Then, we know that $\mathrm{Hom}_R(M, Z)$ is an S^{op}-module and that $Z \otimes_S N$ is an R-module. We can define a canonical homomorphism of $\mathbb{Z}$-modules,

$$\theta\colon \mathrm{Hom}_R(M, Z) \otimes_S N \longrightarrow \mathrm{Hom}_R(M, Z \otimes_S N).$$

For this, for every $n \in N$ and $u \in \mathrm{Hom}_R(M, Z)$, consider the map from M to $Z \otimes_S N$ given by

$$\theta'(u, n)\colon\ m \mapsto u(m) \otimes n.$$

The reader will check (DX) that $\theta'(u, n)$ is R-linear and that $\theta' \in \mathrm{Bi}_S(\mathrm{Hom}_R(M, Z), N; \mathrm{Hom}_R(M, Z \otimes_S N))$. Therefore, we get the desired homomorphism, θ, such that $\theta(u \otimes n)$ is the R-linear map $\theta'(u, n)$. The following proposition holds:

Proposition 2.48

(i) If N is a projective S-module (resp. a f.g. projective S-module), then the $\mathbb{Z}$-homomorphism, $\theta\colon \mathrm{Hom}_R(M, Z) \otimes_S N \longrightarrow \mathrm{Hom}_R(M, Z \otimes_S N)$, is injective (resp. bijective).

(ii) If M is a f.g. projective R-module, then the $\mathbb{Z}$-homomorphism, θ, is bijective.

Proof. In both cases, the proof reduces to the case where M (resp. N) is a free module, and it proceeds by induction on the number of basis vectors in the case where the free module is f.g. (DX). □

The following special case is of special interest: $R = S$ and $Z = R$. In this case, $\mathrm{Hom}_R(M, R) = M^D$, the dual of M, and the $\mathbb{Z}$-homomorphism, θ, becomes

$$\theta\colon M^D \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N),$$

where $\theta(u \otimes n)$ is the R-linear map, $m \mapsto u(m)n$.

Corollary 2.49 *Assume that M and N are R-modules.*

(i) If N is a projective R-module (resp. a f.g. projective R-module), then the $\mathbb{Z}$-homomorphism, $\theta\colon M^D \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N)$, is injective (resp. bijective).

(ii) If M is a f.g. projective R-module, then the $\mathbb{Z}$-homomorphism, θ, is bijective.

If the R-module, N, is also an S^{op}-module, then θ is S^{op}-linear. Similarly, if the R-module, M, is also an S^{op}-module, then θ is S-linear. Furthermore, if M is an R^{op}-module (and N is an R-module), then we obtain a canonical $\mathbb{Z}$-homomorphism,

$$\theta\colon M^{DD} \otimes_R N \longrightarrow \mathrm{Hom}_R(M^D, N).$$

Using the canonical homomorphism, $M \longrightarrow M^{DD}$, we get a canonical homomorphism

$$\theta'\colon M \otimes_R N \longrightarrow \mathrm{Hom}_R(M^D, N).$$

Again, if M is a f.g. projective R^{op}-module, then the map θ' is bijective (DX).

Some (very) important algebras:

Suppose that M is both an R and an R^{op}-module, and that $R \in \mathrm{RNG}$. We also assume, as usual, that M is a bimodule, i.e., $(\rho m)\sigma = \rho(m\sigma)$. Then, $M \otimes_R M$ is again a bimodule, so we can form $M \otimes_R M \otimes_R M$, etc. Define $\mathcal{T}_j(M)$ (also denoted $M^{\otimes j}$) by $\mathcal{T}_0(M) = R$, $\mathcal{T}_1(M) = M$, and

$$\mathcal{T}_j(M) = \underbrace{M \otimes_R \cdots \otimes_R M}_{j}, \quad \text{if } j \geq 2.$$

Then, form

$$\mathcal{T}(M) = \coprod_{j \geq 0} \mathcal{T}_j(M) = \coprod_{j \geq 0} M^{\otimes j}.$$

We can make $\mathcal{T}(M)$ into a ring, by concatenation. Define the map $M^r \times M^s \longrightarrow \mathcal{T}_{r+s}(M)$, by

$$\langle (m_1, \ldots, m_r), (n_1, \ldots, n_s) \rangle \mapsto m_1 \otimes \cdots \otimes m_r \otimes n_1 \otimes \cdots \otimes n_s.$$

This map is bilinear in the pair $\langle(\text{r}-\text{tuple}), (\text{s}-\text{tuple})\rangle$ and so, it is multilinear in all the variables. Thus, we get a map $\mathcal{T}_r(M) \otimes_R \mathcal{T}_s(M) \longrightarrow \mathcal{T}_{r+s}(M)$. Therefore, $\mathcal{T}(M)$ is an R, R^{op}-algebra called the *tensor algebra of M*.

If Z is an R-algebra, denote by (Z) the object Z considered just as an R-module (i.e., $Z \rightsquigarrow (Z)$ is the partial stripping functor from R-alg to $\mathcal{M}\text{od}(R)$.)

Proposition 2.50 *There is a natural, functorial isomorphism*

$$\text{Hom}_{R\text{-alg}}(\mathcal{T}(M), Z) \cong \text{Hom}_{\mathcal{M}\text{od}(R)}(M, (Z)),$$

for every R-algebra, Z. That is, the functor $M \rightsquigarrow \mathcal{T}(M)$ is the left-adjoint of $Z \rightsquigarrow (Z)$.

Proof. Given $\varphi \in \text{Hom}_{R\text{-alg}}(\mathcal{T}(M), Z)$, look at $\varphi \restriction \mathcal{T}_1(M) = \varphi \restriction M$. Observe that $\varphi \restriction M \in \text{Hom}_{\mathcal{M}\text{od}(R)}(M, (Z))$, and clearly, as M generates $\mathcal{T}(M)$, the map φ is determined by $\varphi \restriction M$. We get a functorial and injective map $\text{Hom}_{R\text{-alg}}(\mathcal{T}(M), Z) \longrightarrow \text{Hom}_{\mathcal{M}\text{od}(R)}(M, (Z))$. Say $\psi\colon M \to (Z)$, pick $(m_1, \ldots, m_d) \in M^d$ and form

$$\widetilde{\psi}(m_1, \ldots, m_d) = \psi(m_1) \cdots \psi(m_d).$$

This map is R-multilinear in the m_j's and has values in Z; it gives a map

$$\Xi_d(\psi)\colon \underbrace{M \otimes_R \cdots \otimes_R M}_{d} \longrightarrow Z,$$

and so, we get a map $\Xi(\psi)\colon \mathcal{T}(M) \longrightarrow Z$. It is easy to check that $\varphi \mapsto \varphi \restriction M$ and $\psi \mapsto \Xi(\psi)$ are inverse functorial maps. $\square$

In $\mathcal{T}(M)$, look at the two-sided ideal generated by elements

$$(m \otimes_R n) - (n \otimes_R m),$$

call it $\mathfrak{I}$. Now, $\mathcal{T}$ is a *graded ring*, i.e., it is a coproduct, $\coprod_{j\geq 0} \mathcal{T}_j(M)$, of R-modules and multiplication obeys:

$$\mathcal{T}_j(M) \otimes_R \mathcal{T}_l(M) \subseteq \mathcal{T}_{j+l}(M).$$

The ideal, $\mathfrak{I}$, is a *homogeneous ideal*, which means that

$$\mathfrak{I} = \coprod_{j\geq 0} \mathfrak{I} \cap \mathcal{T}_j(M).$$

To see this, we will in fact prove more:

Proposition 2.51 *Suppose $R = \coprod_{n\geq 0} R_n$ is a graded ring and $\mathfrak{I}$ is a two-sided ideal generated by homogeneous elements $\{r_\alpha\}_{\alpha\in\Lambda}$ (i.e., $r_\alpha \in R_{d_\alpha}$, for some d_α). Then, $\mathfrak{I}$ is a homogeneous ideal. Moreover, the ring, $R/\mathfrak{I}$, is again graded and $R \longrightarrow R/\mathfrak{I}$ preserves degrees.*

Proof. Pick $\xi \in \mathfrak{I}$, then $\xi = \sum_\alpha \rho_\alpha r_\alpha$ and each ρ_α is of the form

$$\rho_\alpha = \sum_{n=0}^{\infty} \rho_{\alpha,n}, \quad \text{where } \rho_{\alpha,n} \in R_n,$$

all the sums involved being, of course, finite. So, we have

$$\xi = \sum_\alpha \sum_{n=0}^{\infty} \rho_{\alpha,n} r_\alpha;$$

moreover, $\rho_{\alpha,n} r_\alpha \in R_{n+d_\alpha}$ and $\rho_{\alpha,n} r_\alpha \in \mathfrak{I}$. As $\mathfrak{I}$ is a 2-sided ideal, the same argument works for $\xi = \sum_\alpha r_\alpha \rho_\alpha$. It follows that

$$\mathfrak{I} = \coprod_{n \geq 0} \mathfrak{I} \cap R_n,$$

and $\mathfrak{I}$ is homogeneous.

Write $\overline{R}$ for $R/\mathfrak{I}$, and let $\overline{R}_n$ be the image of R_n under the homomorphism $\rho \mapsto \overline{\rho}$. Then,

$$\overline{R} = \Big(\coprod_n R_n\Big) / \Big(\coprod_n \mathfrak{I} \cap R_n\Big) \cong \coprod R_n/(\mathfrak{I} \cap R_n).$$

But, $\overline{R}_n = R_n/(\mathfrak{I} \cap R_n)$, so we are done. $\square$

In $\mathcal{T}(M)$, which is graded by the $\mathcal{T}_n(M)$, we have the two 2-sided ideals: $\mathfrak{I}$, the 2-sided ideal generated by the homogeneous elements (of degree 2)

$$m \otimes n - n \otimes m,$$

and $\mathcal{K}$, the 2-sided ideal generated by the homogeneous elements

$$m \otimes m \quad \text{and} \quad m \otimes n + n \otimes m.$$

Both $\mathfrak{I}$ and $\mathcal{K}$ are homogeneous ideals, and by the proposition, $\mathcal{T}(M)/\mathfrak{I}$ and $\mathcal{T}(M)/\mathcal{K}$ are graded rings.

Remark: For $\mathcal{K}$, look at

$$(m+n) \otimes (m+n) = m \otimes m + n \otimes n + m \otimes n + n \otimes m.$$

We deduce that if $m \otimes m \in \mathcal{K}$ for all m, then $m \otimes n + n \otimes m \in \mathcal{K}$ for all m and n. The converse is true if 2 is invertible.

We define $\mathrm{Sym}(M)$, the *symmetric algebra of M* to be $\mathcal{T}/\mathfrak{I}$ and set $m \cdot n =$ image of $m \otimes n$ in $\mathrm{Sym}(M)$. The module $\mathrm{Sym}_j(M)$ is called the *j-th symmetric power of M*. Similarly, $\bigwedge(M) = \mathcal{T}/\mathcal{K}$ is the *exterior algebra of M*, and we set $m \wedge n =$ image of $m \otimes n$ in $\bigwedge(M)$. The module $\bigwedge^j(M)$ is called the *j-th exterior power of M*.

Observe that $m \cdot n = n \cdot m$ in $\mathrm{Sym}(M)$ and $m \wedge n = -n \wedge m$ in $\bigwedge(M)$, for all $m, n \in M$. Of course, $m \wedge m = 0$, for all $m \in M$. Further, $\mathrm{Sym}(M)$ is a commutative ring. However, we can have $\omega \wedge \omega \neq 0$ in $\bigwedge M$; for this, see the remark before Definition 2.8.

The algebras $\mathrm{Sym}(M)$ and $\bigwedge(M)$ are $\mathbb{Z}$-algebras only, even if M is an R-bimodule, unless R is commutative, and then they are R-algebras.

Why?

We know that $r(m \otimes n) = (rm \otimes n)$ in $\mathcal{T}(M)$. But in $\mathrm{Sym}(M)$, we would have (writing $=$ for equivalence mod $\mathfrak{I}$)

$$\begin{aligned} r(m \otimes n) &= (rm) \otimes n \\ &= n \otimes (rm) \\ &= (nr) \otimes m \\ &= m \otimes (nr) \\ &= (m \otimes n)r. \end{aligned}$$

Then, for any $r, s \in R$, we would have

$$\begin{aligned}(rs)(m \otimes n) &= r(s(m \otimes n)) \\ &= r((m \otimes n)s) \\ &= r(m \otimes (ns)) \\ &= (m \otimes (ns))r \\ &= (m \otimes n)(sr).\end{aligned}$$

But, $(sr)(m \otimes n) = (m \otimes n)(sr)$, and so, we would get

$$(rs)(m \otimes n) = (sr)(m \otimes n), \quad \text{for all } r, s \in R.$$

So, if we insist that $\mathrm{Sym}(M)$ and $\bigwedge(M)$ be R-algebras, then R must act as if it were commutative, i.e., the 2-sided ideal, $\mathfrak{M}$, generated by the elements $rs - sr$ $(= [r, s])$ annihilates both our algebras. Yet $R/\mathfrak{M}$ might be the 0-ring. However, in the commutative case, no problem arises.

Proposition 2.52 *Suppose M is an R-bimodule and as R-module it is finitely generated by $e_1, \ldots, e_r$. Then, $\bigwedge^s M = (0)$ if $s > r$.*

Proof. Note that for any $\rho \in M$ and any e_j, we have $e_j\rho \in M$, and so,

$$e_j\rho = \sum_i \lambda_i e_i, \quad \text{for some } \lambda_i\text{'s},$$

in other words, $e_j\rho$ is some linear combination of the e_i's. Elements of $\bigwedge^2 M$ are sums

$$\begin{aligned}\sum_{\beta,\gamma} m_\beta \wedge m_\gamma &= \sum_{\beta,\gamma}\Big(\sum_i \lambda_i^{(\beta)} e_i\Big) \wedge \Big(\sum_j \mu_j^{(\gamma)} e_j\Big) \\ &= \sum_{\beta,\gamma}\sum_{i,j} \lambda_i^{(\beta)}(e_i \wedge \mu_j^{(\gamma)} e_j) \\ &= \sum_{\beta,\gamma}\sum_{i,j} \lambda_i^{(\beta)}(e_i\mu_j^{(\gamma)} \wedge e_j) \\ &= \sum_{l,m} \rho_{lm}(e_l \wedge e_m),\end{aligned}$$

for some ρ_{lm}. An obvious induction shows that $\bigwedge^s M$ is generated by elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_s}$. There are only r distinct e_i's and there are s of the e_i's in our wedge generators; thus, some e_i occurs twice, that is, we have

$$e_{i_1} \wedge \cdots \wedge e_{i_s} = e_{i_1} \wedge \cdots \wedge e_i \wedge \cdots \wedge e_i \wedge \cdots \wedge e_{i_s}.$$

However, we can repeatedly permute the second occurrence of e_i with the term on its left (switching sign each time), until we get two consecutive occurrences of e_i:

$$e_{i_1} \wedge \cdots \wedge e_{i_s} = \pm e_{i_1} \wedge \cdots \wedge e_i \wedge e_i \wedge \cdots \wedge e_{i_s}.$$

As $e_i \wedge e_i = 0$, we get $e_{i_1} \wedge \cdots \wedge e_{i_s} = 0$, and this for every generator. Therefore, $\bigwedge^s M = (0)$. $\square$

Let us now assume that M is a free R-module with basis $e_1, \ldots, e_n$. What are $\mathcal{T}(M)$, $\mathrm{Sym}(M)$ and $\bigwedge(M)$?

The elements of $\mathcal{T}_r(M)$ are sums of terms of the form $m_1 \otimes \cdots \otimes m_r$. Now, each m_i is expressed uniquely as $m_i = \sum_j \lambda_j e_j$. Therefore, in $\mathcal{T}_r(M)$, elements are unique sums of terms of the form

$$(\mu_1 e_{i_1}) \otimes (\mu_2 e_{i_2}) \otimes \cdots \otimes (\mu_r e_{i_r}),$$

where e_{i_l} might be equal to e_{i_k} with $i_l \neq i_k$. Let X_j be the image of e_j in $\mathcal{T}(M)$. Then, we see that the elements of $\mathcal{T}(M)$ are sums of "funny monomials"

$$\mu_1 X_{i_1} \mu_2 X_{i_2} \cdots \mu_d X_{i_d},$$

and in these monomials, we do not have $X\mu = \mu X$ (in general). In conclusion, the general polynomial ring over R in n variables is equal to $\mathcal{T}\left(\coprod_{j=1}^n R\right)$. If our free module is good (i.e., there exists a basis $e_1, \ldots, e_n$ and $\lambda e_i = e_i \lambda$ for all $\lambda \in R$ and all e_i), then we get our simplified noncommutative polynomial ring $R\langle X_1, \ldots, X_n\rangle$, as in Section 2.2.

For $\mathrm{Sym}\left(\coprod_{j=1}^r R\right)$, where $\coprod_{j=1}^r R$ is good, we just get our polynomial ring $R[X_1, \ldots, X_r]$.

All this presumed that the rank of a free finitely-generated R-module made sense. There are rings where this is false. However, if a ring possesses a homomorphism into a field, then ranks do make sense (DX). Under this assumption and assuming that the free module $M = \coprod_{j=1}^r R$ has a good basis, we can determine the ranks of $\mathcal{T}_d(M)$, $\mathrm{Sym}_d(M)$ and $\bigwedge^d(M)$. Since elements of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_d}, \quad \text{where } \{i_1, \ldots, i_d\} \text{ is any subset of } \{1, \ldots, r\}$$

form a basis of $\mathcal{T}_d(M)$, we get $\mathrm{rk}(\mathcal{T}(M)) = r^d$. Linear independence is reduced to the case where R is a field in virtue of our assumption. Here, it is not very difficult linear algebra to prove linear independence. For example, $M \otimes_k N$ is isomorphic to $\mathrm{Hom}_k(M^D, N)$, say by Corollary 2.49.

Elements of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_d}, \quad \text{where } i_1 \leq i_2 \leq \ldots \leq i_d$$

form a basis of $\mathrm{Sym}_d(M)$, so we get $\mathrm{rk}(\mathrm{Sym}_d(M)) = \binom{r+d-1}{d}$ (DX–The linear algebra is the same as before, only the counting is different). Let us check this formula in some simple cases. For $r = d = 2$, the formula predicts dimension 3; indeed, we have the basis of 3 monomials: X_1^2, X_2^2, X_1X_2. For $r = d = 3$, the formula predicts dimension 10; we have the basis of 10 monomials:

$$X_1^3,\ X_2^3,\ X_3^3,\ X_1^2X_2,\ X_1^2X_3,\ X_2^2X_1,\ X_2^2X_3,\ X_3^2X_1,\ X_3^2X_2,\ X_1X_2X_3.$$

Finally, elements of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_d}, \quad \text{where } i_1 < i_2 < \ldots < i_d$$

form a basis of $\bigwedge^d(M)$, so we get $\dim(\bigwedge^d(M)) = \binom{r}{d}$. Again, linear independence follows from the field case. Here, it will be instructive to make a filtration of $\bigwedge^d M$ in terms of lower wedges of M and $\widetilde{M}$, where $\widetilde{M}$ has rank $r-1$. Then, induction can be used. All this will be left to the reader.

And now, an application to a bit of geometry. Let M be a (smooth) manifold of dimension r. For every $x \in M$, we have the *tangent space* to M at x, denoted $T(M)_x$, a rank r vector space. A basis of this vector space is

$$\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_r},$$

where $X_1, \ldots, X_r$ are local coordinates at $x \in M$. A tangent vector is just

$$\sum_{j=1}^r a_j \frac{\partial}{\partial X_j},$$

the directional derivative w.r.t. the vector $\overrightarrow{v} = (a_1, \ldots, a_r)$. The dual space, $T(M)_x^D$, is called the *cotangent space at* x or the *space of* 1*-forms at* x, and has the dual basis: $dX_1, \ldots, dX_r$, where

$$(dX_i)\left(\frac{\partial}{\partial X_j}\right) = \delta_{ij}.$$

Every element of $T(M)_x^D$ is a 1-form at x, i.e., an expression $\sum_{j=1}^r b_j dX_j$. We have the two vector space families $\bigcup_{x\in M} T(M)_x$ and $\bigcup_{x\in M} T(M)_x^D$. These vector space families are in fact *vector bundles* (DX), called the *tangent bundle*, $T(M)$, and the *cotangent bundle*, $T(M)^D$, respectively.

Say $\varphi\colon M \to N$ is a map of manifolds, then we get a vector space map,

$$D\varphi_x\colon T(M)_x \longrightarrow T(N)_{\varphi(x)}.$$

This map can be defined as follows: For any tangent vector, $\xi \in T(M)_x$, at x, pick a curve through x (defined near x), say $z\colon I \to M$, and having our chosen ξ as tangent vector at $t = 0$ (with $x = z(0)$). Here, I is a small open interval about 0. Then,

$$I \xrightarrow{z} M \xrightarrow{\varphi} N$$

is a curve in N through $\varphi(x)$, and we take the derivative of $\varphi(z(t))$ at $t = 0$ to be our tangent vector $(D\varphi_x)(\xi)$.

By duality, there is a corresponding map $(D\varphi_x)^*\colon T(N)_{\varphi(x)}^D \longrightarrow T(M)_x^D$ called *pull-back* of differential forms. Given any open subset, V, of N, for any section, $\omega \in \Gamma(V, \bigwedge^d T(N)^D)$, by pullback we get the section $\varphi^*\omega \in \Gamma(\varphi^{-1}(V), \bigwedge^d T(M)^D)$. The reader should explicate this map in terms of the local coordinates on V and $\varphi^{-1}(V)$.

Now, consider some section, $\omega \in \Gamma(U, \bigwedge^d T(M)^D)$, where U is an open in M. In local coordinates, ω looks like

$$\sum_{i_1<\cdots<i_d} a(x)dx_{i_1} \wedge \cdots \wedge dx_{i_d}; \quad x \in U.$$

Here, U is a piece of a chart, i.e., there is a diffeomorphism $\varphi\colon V\ (\subseteq \mathbb{R}^r) \xrightarrow{\sim} U$. If $z\colon I\ (\subseteq \mathbb{R}^d) \longrightarrow V$ is a map of a D-disk to V, the composition $\varphi \circ z$ is called an *elementary d-chain* in $U \subseteq M$, and a *d-chain* is a formal $\mathbb{Z}$-combination of elementary d-chains. Then, we have $(\varphi\circ z)^*\omega$, a d-form on I. Hence, by elementary real calculus in several variables,

$$\int_I (\varphi \circ z)^*\omega$$

makes sense. ((DX), compute $(\varphi\circ z)^*\omega$ in local coordinates.) We define the integral of ω over the elementary d-chain $\varphi(z(I))$ by

$$\int_{\varphi(z(I))} \omega = \int_I (\varphi \circ z)^*\omega,$$

and for d-chains, let

$$\int_{d-\text{chain}} \omega = \sum \int_{\text{elem. pieces}} \omega.$$

An elaboration of these simple ideas gives the theory of integration of forms on manifolds.

We also have the theory of determinants. Suppose R is a commutative ring and M is a free module of rank d over R with basis $e_1, \ldots, e_d$. So,

$$M \cong \coprod_{j=1}^d Re_j.$$

Let N be another free module of the same rank with basis $f_1, \ldots, f_d$. Then, a linear map $\varphi \in \mathrm{Hom}_R(M, N)$ gives a matrix in the usual way ($\varphi(e_j)$ as linear combination of the f_i's *is* the j-th column). By functoriality, we get a linear map $\bigwedge^d \varphi\colon \bigwedge^d M \to \bigwedge^d N$. Now, each of $\bigwedge^d M$ and $\bigwedge^d N$ is free of rank 1, and their bases are $e_1 \wedge \cdots \wedge e_d$ and $f_1 \wedge \cdots \wedge f_d$, respectively. Therefore,

$$\Big(\bigwedge^d \varphi\Big)(e_1 \wedge \cdots \wedge e_d) = \lambda(f_1 \wedge \cdots \wedge f_d),$$

for some unique $\lambda \in R$. This unique λ is the *determinant* of φ, by definition. Now, $\left(\bigwedge^d \varphi\right)(e_1 \wedge \cdots \wedge e_d) = \varphi(e_1) \wedge \cdots \wedge \varphi(e_d)$, and so $\det(\varphi)$ is an alternating multilinear map on the columns of the matrix of φ. If Q is yet a third free module of rank d and if $\psi\colon N \to Q$ is an R-linear map and $g_1, \ldots, g_d$ a chosen basis for the module Q, then we find that $\bigwedge^d \psi$ takes $f_1 \wedge \cdots \wedge f_d$ to $\mu(g_1 \wedge \cdots \wedge g_d)$, where $\mu = \det(\psi)$. Since $\bigwedge^d \psi$ is R-linear, it takes $\lambda(f_1 \wedge \cdots \wedge f_d)$ to $\lambda\mu(g_1 \wedge \cdots \wedge g_d)$, and it follows that

$$\det(\psi \circ \varphi) = \mu\lambda = \det(\psi)\det(\varphi).$$

It might appear that $\det(\varphi)$ depends upon our choice of basis, but this is not entirely so. If one has two choices of bases in each of M and N, say $\{e_i\}$ and $\{\widetilde{e_i}\}$; $\{f_j\}$ and $\{\widetilde{f_j}\}$, and if the matrices of the identity transformations $M \longrightarrow M$ and $N \longrightarrow N$ in the basis pairs are the same, then $\det(\varphi)$ is the same whether computed with e's and f's or with $\widetilde{e}$'s and $\widetilde{f}$'s. This situation holds when we identify M and N as same rank free modules, then we have just one pair of bases: The $\{e_i\}$ and the $\{\widetilde{e_i}\}$. The determinant of the endomomorphism $\varphi\colon M \to M$ is then independent of the choice of basis.

If M and N have different ranks, say M has rank r with chosen basis $e_1, \ldots, e_r$ while N has rank s with chosen basis $f_1, \ldots, f_s$, then for any R-linear $\varphi\colon M \to N$, we have the induced map

$$\bigwedge^d \varphi\colon \bigwedge^d M \longrightarrow \bigwedge^d N.$$

Consider $e_{j_1} \wedge \cdots \wedge e_{j_d}$, an element of the induced basis for $\bigwedge^d M$. We apply the map $\bigwedge^d \varphi$ and find

$$\left(\bigwedge^d \varphi\right)(e_{j_1} \wedge \cdots \wedge e_{j_d}) = \sum_{1 \le i_1 < \cdots < i_d \le s} \lambda_{i_1 \ldots i_d}^{j_1 \ldots j_d} f_{i_1} \wedge \cdots \wedge f_{i_d}.$$

The element $\lambda_{i_1 \ldots i_d}^{j_1 \ldots j_d} \in R$ is exactly the $d \times d$ minor from the rows $i_1, \ldots, i_d$ and columns $j_1, \ldots, j_d$ of the matrix of φ in the given bases. So, the $d \times d$ minors form the entries for $\bigwedge^d \varphi$. Projectives being cofactors of free modules allow the definition of determinants of their endomorphisms as well. For this, one must study $\bigwedge^d \left(P \amalg \widetilde{P}\right)$. (DX)

For the next two remarks, assume that $R \in \mathrm{CR}$.

Remarks:

(1) Let Z be a commutative R-algebra. Then, the functor, $Z \rightsquigarrow (Z)\,(= Z$ as R-module), has as left-adjoint in CR the functor $M \rightsquigarrow \mathrm{Sym}_R(M)$:

$$\mathrm{Hom}_{R\text{-alg}}(\mathrm{Sym}_R(M), Z) \xrightarrow{\sim} \mathrm{Hom}_R(M, (Z))$$

is a functorial isomorphism (in M and Z).

(2) An *alternating R-algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded R-algebra (which means that $Z = Z_{\mathrm{even}} \amalg Z_{\mathrm{odd}} = Z_0 \amalg Z_1$, with $Z_i Z_j \subseteq Z_{i+j\ (\mathrm{mod}\ 2)}$), together with the commutativity rule

$$\xi\eta = (-1)^{\deg \xi \cdot \deg \eta} \eta\xi.$$

The left-adjoint property for $\bigwedge M$ is this: The functor $Z \rightsquigarrow (Z_1)$ ($= Z_1$ as R-module, where Z is an alternating R-algebra) has $M \rightsquigarrow \bigwedge M$ as left adjoint, i.e.,

$$\mathrm{Hom}_{\text{alt. } R\text{-alg}}\left(\bigwedge M, Z\right) \xrightarrow{\sim} \mathrm{Hom}_R(M, (Z_1))$$

is a functorial isomorphism (in M and Z).

Remark: If $\omega \in (\bigwedge M)_{\mathrm{even}}$, then $\omega \wedge \omega$ need **not** be zero. In fact, if $\xi \in \bigwedge^p M$ and $\eta \in \bigwedge^q M$, then (DX)

$$\xi \wedge \eta = (-1)^{pq} \eta \wedge \xi.$$

Example: $M = \mathbb{R}^4$, $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ (ω is the standard symplectic form on $\mathbb{R}^4$). We have

$$\omega \wedge \omega = (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) = 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \neq 0.$$

Flat Modules. As with the functor Hom, we single out those modules rendering $\otimes$ an exact functor. Actually, before we study right limits, little of consequence can be done. So, here is an introduction and some first properties; we'll return to flatness in Section 2.8.

Definition 2.8 An R^{op}-module, M, is *flat (over R)* iff the functor $N \rightsquigarrow M \otimes_R N$ is exact. If M is an R-module then M is *flat (over R) iff the functor (on R^{op}-modules) $N \rightsquigarrow N \otimes_R M$ is exact. The module, M, is faithfully flat iff M is flat and $M \otimes_R N = (0)$ (resp. $N \otimes_R M = (0)$) implies $N = (0)$.*

Proposition 2.53 *Say M is an R-module (resp. R^{op}-module) and there is another R-module (resp. R^{op}-module), $\widetilde{M}$, so that $M \amalg \widetilde{M}$ is flat. Then M is flat. Finitely generated free modules are faithfully flat. Finitely generated projective modules are flat. Finite coproducts of flat modules are flat. (The finiteness hypotheses will be removed in Section 2.8, but the proofs require the notion of right limit.)*

Proof. Let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence; we treat the case where M is an R^{op}-module. Let $F = M \amalg \widetilde{M}$. As F is flat, the sequence

$$0 \longrightarrow F \otimes_R N' \longrightarrow F \otimes_R N \quad \text{is exact.}$$

We have the diagram

$$\begin{array}{ccccccc}
 & & M \otimes_R N' & \xrightarrow{\quad\theta\quad} & M \otimes_R N & & \\
 & & \downarrow & & \downarrow & & \\
F \otimes_R N' & \xrightarrow{\cong} & M \otimes_R N' \amalg \widetilde{M} \otimes_R N' & \longrightarrow & M \otimes_R N \amalg \widetilde{M} \otimes_R N & \xleftarrow{\cong} & F \otimes_R N.
\end{array}$$

The bottom horizontal arrow is injective and the vertical arrows are injective too, as we see by tensoring the split exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow \widetilde{M} \longrightarrow 0$$

on the right with N and N'. A trivial diagram chase shows that θ is injective, as contended.

Assume F is free and f.g., that is, $F = \coprod_S R$, where $S \neq \emptyset$ and S is finite. Since $F \otimes_R N \cong \coprod_S N$, we have $F \otimes_R N = (0)$ iff $N = (0)$. If we knew that finite coproducts of flats were flat, all we would need to show is that R itself is flat. But, $R \otimes_R N \cong N$, and so, $R \otimes_R -$ is exact.

Let M and $\widetilde{M}$ be flat and consider their coproduct, $F = M \amalg \widetilde{M}$. Then, for any exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

the maps $f \colon M \otimes_R N' \to M \otimes_R N$ and $g \colon \widetilde{M} \otimes_R N' \to \widetilde{M} \otimes_R N$ are injective, as M and $\widetilde{M}$ are flat. Since the coproduct functor is exact, $f \amalg g$ is injective and so

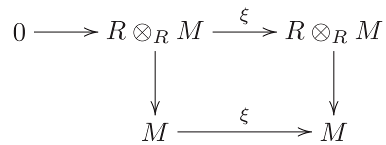
$$(M \otimes_R N') \amalg (\widetilde{M} \otimes_R N') \cong F \otimes_R N' \longrightarrow F \otimes_R N \cong (M \otimes_R N) \amalg (\widetilde{M} \otimes_R N)$$

is injective as well, which proves that F is flat.

If P is projective and f.g., then $P \amalg \widetilde{P} \cong F$, for some module $\widetilde{P}$ and some f.g. free module, F. The first part of the proof shows that P is flat. $\square$

Proposition 2.54 *If $R \in$ CR is an integral domain (or $R \in$ RNG has no zero divisors) then every flat module is torsion-free. The converse is true if R is a P.I.D. (the proof will be given in Section 2.8).*

Proof. If $\xi \in R$, then $0 \longrightarrow R \stackrel{\xi}{\longrightarrow} R$ is an injective $R^{\rm op}$-homomorphism ($(\xi m)\rho = \xi(m\rho)$). The diagram



commutes, the vertical arrows are isomorphisms, and the upper row is exact, since M is flat. This shows that $m \mapsto \xi m$ is injective; so, if $\xi m = 0$, then $m = 0$. □

Remark: The module $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. However, $\mathbb{Q}$ is **not** free, **not** projective (DX) and **not** faithfully flat ($\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = (0)$).

2.7 Limit Processes in Algebra

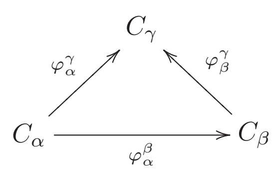
Let Λ be a partially ordered set (with partial order $\leq$) and assume Λ has the *Moore–Smith property* (Λ is a *directed* set), which means that for all $\alpha, \beta \in \Lambda$, there is some $\gamma \in \Lambda$ so that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Examples of Directed Sets: (1) Let X be a topological space, and pick $x \in X$; take $\Lambda = \{U \mid (1)\, U \text{ open in } X; (2)\, x \in U\}$, with $U \leq V$ iff $V \subseteq U$.

(2) $\Lambda = \mathbb{N}$, and $n \leq m$ iff $n \mid m$ (Artin ordering).

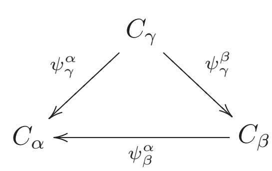
To introduce right and left limits, we consider the following set-up: We have a category, $\mathcal{C}$, a collection of objects of $\mathcal{C}$ indexed by Λ, say C_α. Consider the two conditions (R) and (L) stated below:

(R) For all $\alpha \leq \beta$, there is a morphism, $\varphi_\alpha^\beta \colon C_\alpha \to C_\beta$, and there is compatibility: For all $\alpha \leq \beta \leq \gamma$, the diagram



commutes and $\varphi_\alpha^\alpha = \mathrm{id}_{C_\alpha}$.

(L) For all $\alpha \leq \beta$, there is a morphism, $\psi_\beta^\alpha \colon C_\beta \to C_\alpha$, and there is compatibility: For all $\alpha \leq \beta \leq \gamma$, the diagram



commutes and $\psi_\alpha^\alpha = \mathrm{id}_{C_\alpha}$.

Definition 2.9 A *right (direct, inductive) mapping family, $(C_\alpha, \varphi_\alpha^\beta)$, of $\mathcal{C}$* is a family of objects, C_α, and *morphisms*, φ_α^β, satisfying axiom (R). *Mutatis mutandis* for a *left (inverse, projective) mapping family*, $(C_\alpha, \psi_\beta^\alpha)$ and axiom (L).

Examples of Right and Left Mapping Families:

(1L) Let $\Lambda = \mathbb{N}$ with the usual ordering, $\mathcal{C} = \mathcal{A}\text{b}$ and $C_n = \mathbb{Z}$. Pick a prime, p; for $m \le n$, define $\psi_n^m \colon \mathbb{Z} \to \mathbb{Z}$ as multiplication by p^{n-m}.

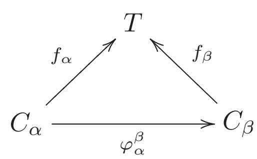
(1R) Same Λ, same $\mathcal{C}$, same C_n, and $\varphi_m^n \colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by p^{n-m}.

(2L) Same Λ, Artin ordering, same $\mathcal{C}$, same C_n. If $n \le m$, then $n \mid m$, so $m\mathbb{Z} \subseteq n\mathbb{Z}$, define $\psi_m^n \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ as the projection map.

(2R) Same Λ, Artin ordering, same $\mathcal{C}$, $C_n = \mathbb{Z}/n\mathbb{Z}$. If $n \le m$, then $r = m/n \in \mathbb{Z}$, define $\varphi_n^m \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ as multiplication by r.

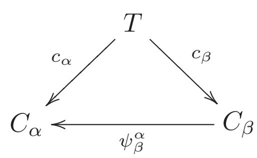
Look at the functor (from $\mathcal{C}$ to $\mathcal{S}$ets)

$$T \rightsquigarrow \left\{ (f_\alpha \colon C_\alpha \longrightarrow T)_\alpha \;\middle|\; \text{[diagram] commutes whenever } \alpha \le \beta \right\},$$



denoted $\varinjlim_\alpha (C_\alpha, \varphi_\alpha^\beta)(T)$, and the cofunctor (from $\mathcal{C}$ to $\mathcal{S}$ets)

$$T \rightsquigarrow \left\{ (g_\alpha \colon T \longrightarrow C_\alpha)_\alpha \;\middle|\; \text{[diagram] commutes whenever } \alpha \le \beta \right\},$$

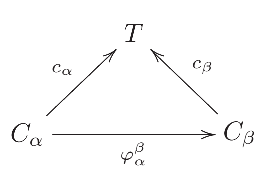


denoted $\varprojlim_\beta (C_\beta, \psi_\beta^\alpha)(T)$.

Question: *Are either (or both) of these representable?*

Definition 2.10 The *right (direct, inductive) limit* of a right mapping family, $(C_\alpha, \varphi_\alpha^\beta)$, is the pair, $(C, \{c_\alpha\})$, representing the functor $\varinjlim_\alpha (C_\alpha, \varphi_\alpha^\beta)$ and is denoted $\varinjlim_\alpha (C_\alpha, \varphi_\alpha^\beta)$. The *left (inverse, projective) limit* of a left mapping family, $(C_\beta, \psi_\beta^\alpha)$, is the pair, $(C, \{c_\beta\})$, representing the functor $\varprojlim_\beta (C_\beta, \psi_\beta^\alpha)$, denoted $\varprojlim_\beta (C_\beta, \psi_\beta^\alpha)$.

Let us explicate this definition. First, consider right mapping families. The tuple $\{c_\alpha\}_\alpha$ is to lie in $\varinjlim_\alpha (C_\alpha, \varphi_\alpha^\beta)(C)$, the set of tuples of morphisms, $c_\alpha \colon C_\alpha \to C$, so that the diagram



commutes whenever $\alpha \le \beta$. We seek an object, $C \in \mathcal{C}$, and a family of morphisms, $c_\alpha \colon C_\alpha \to C$, so that

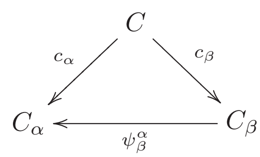
$$\mathrm{Hom}_{\mathcal{C}}(C, T) \cong \varinjlim_\alpha (C_\alpha, \varphi_\alpha^\beta)(T),$$

for every $T \in \mathcal{C}$, *via* the isomorphism $u \mapsto \{u \circ c_\alpha\}_\alpha$. Thus, the above functorial isomorphism says that for every family of morphisms, $\{f_\alpha \colon C_\alpha \to T\}_\alpha \in \underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\,(C_\alpha, \varphi_\alpha^\beta)(T)$, there is a *unique* morphism, $u \colon C \to T$, so that

$$f_\alpha = u \circ c_\alpha, \quad \text{for all } \alpha \in \Lambda.$$

This is the *universal mapping property* of $\underset{\alpha}{\varinjlim}\, C_\alpha$.

Next, consider left mapping families. This time, the tuple $\{c_\beta\}_\beta$ is to lie in $\underset{\beta}{\underleftarrow{\mathrm{Lim}}}\,(C_\beta, \psi_\beta^\alpha)(C)$, the set of tuples of morphisms, $c_\beta \colon C \to C_\beta$, so that the diagram



commutes whenever $\alpha \le \beta$. We seek an object, $C \in \mathcal{C}$, and a family of morphisms, $c_\beta \colon C \to C_\beta$, so that

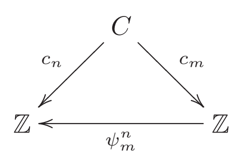
$$\mathrm{Hom}_{\mathcal{C}}(T, C) \cong \underset{\beta}{\underleftarrow{\mathrm{Lim}}}\,(C_\beta, \psi_\beta^\alpha)(T),$$

for every $T \in \mathcal{C}$, *via* the isomorphism $u \mapsto \{c_\beta \circ u\}_\beta$. The *universal mapping property* of $\underset{\alpha}{\varprojlim}\, C_\alpha$ is that for every family of morphisms, $\{g_\alpha \colon T \to C_\alpha\}_\alpha \in \underset{\beta}{\underleftarrow{\mathrm{Lim}}}\,(C_\beta, \psi_\beta^\alpha)(T)$, there is a *unique* morphism, $u \colon T \to C$, so that

$$g_\alpha = c_\alpha \circ u, \quad \text{for all } \alpha \in \Lambda.$$

Remark: A right (resp. left) mapping family in $\mathcal{C}$ is the same as a left (resp. right) mapping family in the dual category $\mathcal{C}^D$. Thus, $\underset{\alpha}{\varinjlim}\,(C_\alpha)$ exists in $\mathcal{C}$ iff $\underset{\alpha}{\varprojlim}\,(C_\alpha)$ exists in $\mathcal{C}^D$.

Let us examine Example (1L). If we assume that its inverse limit exists, then we can find out what this is. By definition, whenever $n \le m$, the map $\psi_m^n \colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by p^{m-n}. Pick $\xi \in C$, hold n fixed and look at $c_n(\xi) \in \mathbb{Z}$. For all $m \ge n$, the commutativity of the diagram



shows that $p^{m-n} c_m(\xi) = c_n(\xi)$, and so, p^{m-n} divides $c_n(\xi)$ for all $m \ge n$. This can only be true if $c_n \equiv 0$. Therefore, all the maps, c_n, are the zero map. As there is a unique homomorphism from any abelian group, T, to (0) and as the tuple of maps, $\{c_\alpha\}_\alpha$, is the tuple of zero maps, the group (0) with the zero maps is $\underset{\alpha}{\varprojlim}\, C_\alpha$. In fact, this argument with T replacing C *proves* the existence of the left limit for the family (1L) and exhibits it as (0).

Theorem 2.55 *(Existence Theorem) If $\mathcal{C}$ is any one of the categories:* $\mathcal{S}$ets, *Ω-groups (includes R-modules, vector spaces,* $\mathcal{A}$b, $\mathcal{G}$r*), topological spaces, topological groups,* CR, RNG*, then both* $\underset{\alpha}{\underrightarrow{\mathrm{Lim}}}$ *and* $\underset{\alpha}{\underleftarrow{\mathrm{Lim}}}$ *are representable (we say that $\mathcal{C}$ possesses arbitrary right and left limits).*

Proof. We give a complete proof for $\mathcal{S}$ets and indicate the necessary modifications for the other categories. Let Λ be a directed index set.

(1) *Right limits*: For every $\alpha \in \Lambda$, we have a set, S_α, and we have set maps, $\varphi_\alpha^\beta \colon S_\alpha \to S_\beta$, whenever $\alpha \leq \beta$. Let $\mathcal{S} = \bigcup^{\cdot} S_\alpha$, the coproduct of the S_α's in $\mathcal{S}$ets (their disjoint union). Define an equivalence relation on $\mathcal{S}$ as follows: For all $x, y \in \mathcal{S}$,

$$\text{if } x \in S_\alpha \text{ and } y \in S_\beta \text{ then} \quad x \sim y \quad \text{iff} \quad (\exists \gamma \in \Lambda)(\alpha \leq \gamma,\ \beta \leq \gamma)(\varphi_\alpha^\gamma(x) = \varphi_\beta^\gamma(y)).$$

We need to check that $\sim$ is an equivalence relation. It is obvious that $\sim$ is reflexive and symmetric.

Say $x \sim y$ and $y \sim z$. This means that $x \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$ and there exist $\delta_1, \delta_2 \in \Lambda$ so that $\alpha \leq \delta_1$; $\beta \leq \delta_1$; $\beta \leq \delta_2$; $\gamma \leq \delta_2$, and

$$\varphi_\alpha^{\delta_1}(x) = \varphi_\beta^{\delta_1}(y);\ \varphi_\beta^{\delta_2}(y) = \varphi_\gamma^{\delta_2}(z).$$

As Λ is directed, there is some $\delta \in \Lambda$, with $\delta_1 \leq \delta$ and $\delta_2 \leq \delta$; so, we may replace δ_1 and δ_2 by δ. Therefore, $\varphi_\alpha^\delta(x) = \varphi_\gamma^\delta(z)$, and transitivity holds. Let $S = \mathcal{S}/\sim$. We have the maps

$$s_\alpha \colon S_\alpha \longrightarrow \bigcup_\lambda^{\cdot} S_\lambda = \mathcal{S} \xrightarrow{pr} \mathcal{S}/\sim\, = S,$$

and the pair $(S, \{s_\alpha\})$ represents $\varinjlim_\alpha S_\alpha$, as is easily checked.

(2) *Left Limits*: We have sets, S_α, for every $\alpha \in \Lambda$, and maps, $\psi_\beta^\alpha \colon S_\beta \to S_\alpha$. Let

$$P = \Big\{(\xi_\alpha) \in \prod_\alpha S_\alpha \;\Big|\; (\forall \alpha \leq \beta)(\psi_\beta^\alpha(\xi_\beta) = \xi_\alpha)\Big\},$$

be the collection of consistent tuples from the product. The set P might be empty.

We have the maps

$$p_\alpha \colon P \hookrightarrow \prod_\alpha S_\alpha \xrightarrow{pr_\alpha} S_\alpha.$$

The pair $(P, \{p_\alpha\})$ represents the cofunctor $\varprojlim_\alpha S_\alpha$ (DX).

Modifications: Look first at the category of groups (this also works for Ω-groups and rings).

(1′) *Right limits*. Write G_α for each group ($\alpha \in \Lambda$). We claim that $G = \varinjlim_\alpha G_\alpha$ (in $\mathcal{S}$ets) is already a group (etc., in a natural way) and as a group, it represents our functor. All we need to do is to define the group operation on $\varinjlim_\alpha G_\alpha$. If $x, y \in G = \varinjlim_\alpha G_\alpha$, then $x = c_\alpha(\xi)$ and $y = c_\beta(\eta)$, for some $\xi \in G_\alpha$ and some $\eta \in G_\beta$. Since Λ is directed, there is some $\gamma \in \Lambda$ with $\alpha, \beta \leq \gamma$; consider $\xi' = \varphi_\alpha^\gamma(\xi)$ and $\eta' = \varphi_\beta^\gamma(\eta)$. (Obviously, $c_\gamma(\xi') = x$ and $c_\gamma(\eta') = y$.) So, we have $\xi', \eta' \in G_\gamma$, and we set

$$xy = c_\gamma(\xi'\eta').$$

Check (DX) that such a product is well-defined and that G is a group. Also, the maps c_α are group homomorphisms.

The existence of right limits now holds for all the algebraic categories.

Now, consider the category, TOP, of topological spaces. Observe that when each S_α is a topological space, then the disjoint union, $\mathcal{S} = \bigcup^{\cdot} S_\alpha$, is also a topological space (using the disjoint union topology); in fact, it is the coproduct in TOP. Give $S = \mathcal{S}/\sim$ the quotient topology, and then check that the maps s_α are continuous and that $(S, \{s_\alpha\})$ represents $\varinjlim_\alpha S_\alpha$ in TOP.

For the category of topological groups, TOPGR, check that $G = \varinjlim_{\alpha} G_\alpha$ is also a topological space as above and (DX) that the group operations are continuous. Thus, $(G, \{s_\alpha\}_\alpha)$ represents $\underrightarrow{\mathrm{Lim}}_{\alpha} G_\alpha$ in TOPGR.

(2′) *Left Limits*. Again, first assume each G_α is a group and the ψ^α_β are homomorphisms. Check that P (= consistent tuples) is a group (in particular, note that $(1, 1, \ldots, 1, \ldots)$ is consistent so that $P \neq \emptyset$) and that the p_α's are homomorphisms (DX); hence, $(P, \{p_\alpha\})$ represents $\underleftarrow{\mathrm{Lim}}_{\alpha} G_\alpha$. Now, similar reasoning shows left limits exist for all the algebraic categories.

For TOP, we make $\prod_\alpha S_\alpha$ into a topological space with the product topology. Check (DX) that the continuity of the ψ^α_β's implies that P is closed in $\prod_\alpha S_\alpha$. Then, the p_α's are also continuous and $(P, \{p_\alpha\})$ represents $\underleftarrow{\mathrm{Lim}}_{\alpha} S_\alpha$ in TOP.

For TOPGR, similar remarks, as above for TOP and as in the discussion for groups, imply that $(P, \{p_\alpha\})$ represents $\underleftarrow{\mathrm{Lim}}_{\alpha} G_\alpha$ in TOPGR. $\square$

Remark: Say Λ is a directed index set. We can make Λ a category as follows: $\mathcal{O}\mathrm{b}(\Lambda) = \Lambda$, and

$$\mathrm{Hom}(\alpha, \beta) = \begin{cases} \emptyset & \text{if } \alpha \not\leq \beta; \\ \{\cdot\} & \text{if } \alpha \leq \beta. \end{cases}$$

(Here, $\{\cdot\}$ denotes a one-point set.) Given a right mapping family, $(C_\alpha, \varphi^\beta_\alpha)$, where $\varphi^\beta_\alpha \in \mathrm{Hom}_{\mathcal{C}}(C_\alpha, C_\beta)$, we define the functor, RF, by

$$\begin{aligned} \mathrm{RF}(\alpha) &= C_\alpha \\ \mathrm{RF}(\cdot\colon \alpha \to \beta) &= \varphi^\beta_\alpha. \end{aligned}$$

Similarly, there is a one-to-one correspondence between left-mapping families, $(C_\beta, \psi^\alpha_\beta)$, and cofunctors, LF, defined by

$$\begin{aligned} \mathrm{LF}(\alpha) &= C_\alpha \\ \mathrm{LF}(\cdot\colon \alpha \to \beta) &= \psi^\alpha_\beta. \end{aligned}$$

If we now think of RF and LF as "functions" on Λ and view the Moore–Smith property as saying that the α's "grow without bound", then we can interpret $\varinjlim_{\alpha} C_\alpha$ and $\varprojlim_{\alpha} C_\alpha$ as: "limits, as $\alpha \to \infty$, of our 'functions' RF and LF",

$$\varinjlim_{\alpha} C_\alpha = \lim_{\alpha\to\infty} \mathrm{RF}(\alpha) \quad \text{and} \quad \varprojlim_{\alpha} C_\alpha = \lim_{\alpha\to\infty} \mathrm{LF}(\alpha).$$

Indeed, there is a closer analogy. Namely, we are taking the limit of $\mathrm{RF}(\alpha)$ and $\mathrm{LF}(\alpha)$ as *nets* in the sense of general topology.

Say $\Gamma \subseteq \Lambda$ is a subset of our index set, Λ. We say that Γ is *final* in Λ (old terminology, *cofinal*) iff for every $\alpha \in \Lambda$, there is some $\beta \in \Gamma$ with $\alpha \leq \beta$. Check (DX),

$$\varinjlim_{\alpha\in\Gamma} C_\alpha = \varinjlim_{\alpha\in\Lambda} C_\alpha; \qquad \varprojlim_{\alpha\in\Gamma} C_\alpha = \varprojlim_{\alpha\in\Lambda} C_\alpha.$$

Examples of Right and Left Limits:

(1R) Recall that $\Lambda = \mathbb{N}$ with the ordinary ordering, $C_n = \mathbb{Z}$ and for $m \geq n$, φ_n^m is multiplication by p^{m-n}. Consider the isomorphism, $\theta_n \colon \mathbb{Z} \to (1/p^n)\mathbb{Z} \subseteq \mathbb{Q}$, defined by $\theta_n(1) = 1/p^n$. The diagram

$$\begin{array}{ccccc} C_n = \mathbb{Z} & \xrightarrow{\theta_n} & \dfrac{1}{p^n}\mathbb{Z} & \hookrightarrow & \mathbb{Q} \\ \Big\downarrow{\scriptstyle p^{m-n}} & & \Big\downarrow{\scriptstyle \text{incl}} & & \Big\| \\ C_m = \mathbb{Z} & \xrightarrow[\theta_m]{} & \dfrac{1}{p^m}\mathbb{Z} & \hookrightarrow & \mathbb{Q} \end{array}$$

commutes, and so, the direct limit on the left is equal to the direct limit in the middle. There, the direct limit is

$$\varinjlim_m C_m = \left\{ \frac{k}{p^t} \;\middle|\; k \in \mathbb{Z},\ p \nmid k \right\} \subseteq \mathbb{Q}.$$

This subgroup, $\varinjlim\limits_m C_m$, of $\mathbb{Q}$ is usually denoted $\dfrac{1}{p^\infty}\mathbb{Z}$.

Generalization: $\Lambda = \mathbb{N}$, Artin ordering ($n \leq m$ iff $n \mid m$), $C_n = \mathbb{Z}$, and for $n \leq m$, define, $\varphi_n^m =$ multiplication by m/n. We get

$$\varinjlim_n C_n = \mathbb{Q}. \tag{$*$}$$

(2R) What is $\varinjlim\limits_{n|m} \mathbb{Z}/n\mathbb{Z}$? If we observe that $\mathbb{Z}/n\mathbb{Z} \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, by $(*)$, we get

$$\varinjlim_{n|m} \mathbb{Z}/n\mathbb{Z} = \mathbb{Q}/\mathbb{Z}.$$

Say X and Y are topological spaces and pick $x \in X$. Let

$$\Lambda_x = \{U \mid U \text{ open in } X \text{ and } x \in U\};$$

Partially order Λ_x so that $U \leq V$ iff $V \subseteq U$ (usual ordering on Λ_x). Clearly, Λ_x has Moore–Smith. Let

$$\mathcal{C}(U) = \left\{ f \;\middle|\; \begin{array}{l} (1)\ f \colon U \to Y \\ (2)\ f \text{ is continuous on } U \text{ (or perhaps has better properties)} \end{array} \right\}$$

Look at $\varinjlim\limits_{\Lambda_x} \mathcal{C}(U)$, denoted temporarily C_x. We have $\xi \in C_x$ iff there is some open subset, U, of X, with $x \in U$, some continuous function, $f \colon U \to Y$, and ξ is the class of f.

Two functions, $f \colon U \to Y$ and $g \colon V \to Y$, where $U, V \subseteq X$ are open and contain x, give the same ξ iff there is some open, $W \subseteq U \cap V$, with $x \in W$, so that $f \restriction W = g \restriction W$. Therefore, C_x is the set of germs of continuous functions on X at x. (The usual notation for C_x is $\mathcal{O}_{X,x}$.)

(2L) Consider the left limit, $\varprojlim\limits_{n|m} \mathbb{Z}/n\mathbb{Z}$, where $\psi_m^n \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is projection. The elements of $\varprojlim\limits_{n|m} \mathbb{Z}/n\mathbb{Z}$ are tuples, (ξ_n), with $\xi_n \in \mathbb{Z}$, such that

(1) $(\xi_n) = (\eta_n)$ iff $(\forall n)(\xi_n \equiv \eta_n \pmod{n})$ and

(2) (consistency): If $n \mid m$, then $\xi_m \equiv \xi_n \pmod{n}$.

We obtain a new object, denoted $\widehat{\mathbb{Z}}$. We have an injective map, $\mathbb{Z} \longrightarrow \widehat{\mathbb{Z}}$, given by $n \mapsto (n, n, \ldots, n, \ldots)$. You should check that the following two statements are equivalent:

(1) Chinese Remainder Theorem.

(2) $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$.

Proposition 2.56 *Say $C = \varinjlim_{\alpha} C_\alpha$ and let $x \in C_\alpha$ and $y \in C_\beta$, with $c_\alpha(x) = c_\beta(y)$. Then, there is some $\gamma \geq \alpha, \beta$, so that $\varphi_\alpha^\gamma(x) = \varphi_\beta^\gamma(y)$. In particular, if all the φ_α^β are injections, so are the canonical maps, c_α.*

Proof. Clear. □

Corollary 2.57 *Say $\mathcal{C} = \Omega$-modules and each C_α is Ω-torsion-free. Then, $\varinjlim_{\alpha} C_\alpha$ is torsion-free.*

Proof. Pick $x \in C = \varinjlim_{\alpha} C_\alpha$; $\lambda \in \Omega$, with $\lambda \neq 0$. Then, $\lambda x = \lambda c_\alpha(x_\alpha)$, for some α and some $x_\alpha \in C_\alpha$. So, $0 = \lambda x = c_\alpha(\lambda x_\alpha)$ implies that there is some $\gamma \geq \alpha$, with $\varphi_\alpha^\gamma(\lambda x_\alpha) = 0$. Consequently, $\lambda\varphi_\alpha^\gamma(x_\alpha) = \lambda x_\gamma = 0$. But C_γ is torsion-free, so $x_\gamma = 0$. Therefore, $x = c_\alpha(x_\alpha) = c_\gamma(x_\gamma) = 0$. This proves that C is torsion-free. □

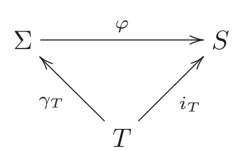
Corollary 2.58 *Say $\mathcal{C} = \Omega$-modules and each C_α is Ω-torsion. Then, $\varinjlim_{\alpha} C_\alpha$ is torsion.*

Proof. If $x \in C$, then there is some α and some $x_\alpha \in C_\alpha$, with $c_\alpha(x_\alpha) = x$. But, there is some $\lambda \in \Omega$, with $\lambda \neq 0$, so that $\lambda x_\alpha = 0$, since C_α is torsion. So, $\lambda x = \lambda c_\alpha(x_\alpha) = c_\alpha(\lambda x_\alpha) = 0$. □

Proposition 2.59 *Let Λ be an index set and $\mathcal{C} = \mathcal{S}$ets. Then, every set is the right-limit of its finite subsets (under inclusion). The same conclusion holds if $\mathcal{C} = \mathcal{G}$r, Ω-groups,* RNG*, then each object of $\mathcal{C}$ is equal to the right limit of its finitely generated subobjects.*

Proof. Let $\Lambda = \{T \subseteq S \mid T \text{ finite}\}$. Order Λ, *via* $T \leq W$ iff $T \subseteq W$. Clearly, Λ has Moore–Smith. Let $\Sigma = \varinjlim_{T \in \Lambda} T$.

For a given $T \in \Lambda$, we have an injective map, $i_T \colon T \hookrightarrow S$. Hence, by the universal mapping property, these maps factor through the canonical maps, $\gamma_T \colon T \to \Sigma$, *via* a fixed map, $\varphi \colon \Sigma \to S$:



Pick some $\xi \in S$. Then, $\{\xi\} \in \Lambda$; so we get a map, $\gamma_{\{\xi\}} \colon \{\xi\} \to \Sigma$. Let $\psi(\xi) = \gamma_{\{\xi\}}(\xi) \in \Sigma$. This gives a map, $\psi \colon S \to \Sigma$. Check (DX), φ and ψ are inverse maps.

Modifications: $\Lambda = \{T \subseteq S \mid T \text{ is a finitely generated subobject of } S\}$ and proceed analogously. □

Corollary 2.60 *An abelian group is torsion iff it is a right-limit of finite abelian groups.*

Corollary 2.61 *Say $\mathcal{C}$ is a category with finite coproducts (or finite products). If $\mathcal{C}$ has right limits (resp. left limits) then $\mathcal{C}$ has arbitrary coproducts (resp. arbitrary products).*

Proof. Cf. Problem 62. □

Proposition 2.62 *Say $\{G_\alpha\}_\alpha$ is a left-mapping family of finite groups (not necessarily abelian). Then, the left limit, $\varprojlim_\alpha G_\alpha = G$, is a compact topological group. (Such a G is called a profinite group.) Similarly, if the G_α are compact topological groups and form a left-mapping family with continuous homomorphisms, then $\varprojlim_\alpha G_\alpha = G$ is a compact topological group.*

Proof. Observe that the second statement implies the first. Now, G is the group of consistent tuples in $\prod_\alpha G_\alpha$. By Tychonov's theorem, $\prod_\alpha G_\alpha$ is compact. As the ψ_β^α are continuous, the subgroup of consistent tuples is *closed*; therefore, this subgroup is compact. $\square$

It follows from Proposition 2.62 that $\widehat{\mathbb{Z}}$ is compact.

2.8 Flat Modules (Again)

Proposition 2.63 *Say $\{\Omega_\alpha\}_\alpha$ is a right-mapping family of rings, $\{M_\alpha\}_\alpha$, $\{N_\alpha\}_\alpha$ are "right-mapping families" of $\Omega_\alpha^{\mathrm{op}}$ (resp. Ω_α)-modules, then $\{M_\alpha \otimes_{\Omega_\alpha} N_\alpha\}_\alpha$ forms a right-mapping family (in $\mathcal{A}\mathrm{b}$) and*

$$\varinjlim_\alpha (M_\alpha \otimes_{\Omega_\alpha} N_\alpha) = \Big(\varinjlim_\alpha M_\alpha\Big) \otimes_{\varinjlim_\alpha \Omega_\alpha} \Big(\varinjlim_\alpha N_\alpha\Big).$$

Proof. The hypothesis (within quotes) means that for all $\alpha \leq \beta$, we have

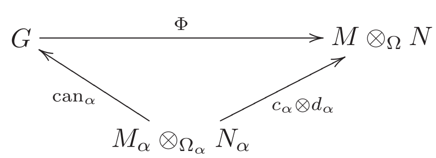
$$\psi_\alpha^\beta(\lambda_\alpha n_\alpha) = \theta_\alpha^\beta(\lambda_\alpha)\psi_\alpha^\beta(n_\alpha), \quad \text{for all } \lambda_\alpha \in \Omega_\alpha \text{ and all } n_\alpha \in N_\alpha,$$

where $\psi_\alpha^\beta \colon N_\alpha \to N_\beta$ and $\theta_\alpha^\beta \colon \Omega_\alpha \to \Omega_\beta$, and similarly with the M_α's.

Let $M = \varinjlim_\alpha M_\alpha$; $N = \varinjlim_\alpha N_\alpha$; $\Omega = \varinjlim_\alpha \Omega_\alpha$ and $G = \varinjlim_\alpha (M_\alpha \otimes_{\Omega_\alpha} N_\alpha)$. Write $c_\alpha \colon M_\alpha \to M$; $d_\alpha \colon N_\alpha \to N$ and $t_\alpha \colon \Omega_\alpha \to \Omega$, for the canonical maps. We have the maps

$$c_\alpha \otimes d_\alpha \colon M_\alpha \otimes_{\Omega_\alpha} N_\alpha \longrightarrow M \otimes_\Omega N,$$

hence, by the universal mapping property of right limits, there is a unique map, $\Phi \colon G \to M \otimes_\Omega N$, so that the following diagram commutes for every α:



We also need a map, $M \otimes_\Omega N \longrightarrow G$. Pick $m \in M$ and $n \in N$, since the index set is directed we may assume that there is some α so that $m = c_\alpha(m_\alpha)$ and $n = d_\alpha(n_\alpha)$. Thus, we have $m_\alpha \otimes_{\Omega_\alpha} n_\alpha \in M_\alpha \otimes_{\Omega_\alpha} N_\alpha$ and so, $\mathrm{can}_\alpha(m_\alpha \otimes_{\Omega_\alpha} n_\alpha) \in G$. Define Ψ by

$$\Psi(m, n) = \mathrm{can}_\alpha(m_\alpha \otimes_{\Omega_\alpha} n_\alpha).$$

Check (DX) that

(1) Ψ is well-defined,

(2) Ψ is bilinear; thus, by the universal mapping property of tensor, there is a map, $\Psi \colon M \otimes_\Omega N \to G$,

(3) Φ and Ψ are inverse homomorphisms. $\square$

Proposition 2.64 *Suppose $\mathcal{C} = \mathcal{M}\mathrm{od}(\Omega)$ and N'_α, N_α, N''_α, are all right-mapping families of Ω-modules. If for every α, the sequence*

$$0 \longrightarrow N'_\alpha \longrightarrow N_\alpha \longrightarrow N''_\alpha \longrightarrow 0 \quad \textit{is exact,}$$

then the sequence

$$0 \longrightarrow \varinjlim_\alpha N'_\alpha \longrightarrow \varinjlim_\alpha N_\alpha \longrightarrow \varinjlim_\alpha N''_\alpha \longrightarrow 0 \quad \textit{is again exact.}$$

Proof. (DX) □

Corollary 2.65 *The right-limit of flat modules is flat.*

Proof. The operation $\varinjlim_\alpha$ commutes with tensor and preserves exactness, as shown above. □

Corollary 2.66 *Tensor product commutes with arbitrary coproducts. An arbitrary coproduct of flat modules is flat.*

Proof. Look at $\coprod_{\alpha\in S} M_\alpha$. We know from the Problems that $\coprod_{\alpha\in S} M_\alpha = \varinjlim_T M_T$, where $T \subseteq S$, with T finite and $M_T = \coprod_{\beta\in T} M_\beta$. So, given N, we have

$$\begin{aligned} N \otimes_\Omega \Big(\coprod_S M_\alpha\Big) &= N \otimes_\Omega \varinjlim_T M_T \\ &= \varinjlim_T (N \otimes_\Omega M_T) \\ &= \varinjlim_T \coprod_{\beta\in T} (N \otimes_\Omega M_\beta) \\ &= \coprod_{\beta\in S} (N \otimes_\Omega M_\beta). \end{aligned}$$

The second statement follows from Corollary 2.65 and the fact that finite coproducts of flat modules are flat (Proposition 2.53). □

Remark: Corollary 2.66 extends the last part of Proposition 2.44 that only asserts that tensor commutes with finite coproducts. It also proves that Proposition 2.53 holds for arbitrary modules, not just f.g. modules. Thus, free modules are flat and so, projective modules are flat, too.

Proposition 2.67 *Say Ω is a ring and M is an Ω^{op}-module (resp. Ω-module). Then, M is flat iff for every exact sequence*

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of Ω (resp. Ω^{op})-modules in which all three modules are f.g., the induced sequence

$$0 \longrightarrow M \otimes_\Omega N' \longrightarrow M \otimes_\Omega N \longrightarrow M \otimes_\Omega N'' \longrightarrow 0$$

(*resp.* $$0 \longrightarrow N' \otimes_\Omega M \longrightarrow N \otimes_\Omega M \longrightarrow N'' \otimes_\Omega M \longrightarrow 0)$$

remains exact.

Proof. Given

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

an arbitrary exact sequence of Ω-modules, write $N = \varinjlim_\alpha N_\alpha$, where the N_α's are f.g. submodules of N. Let N''_α be the image of N_α in N''. So, N''_α is f.g., too. We get the exact sequence

$$0 \longrightarrow N' \cap N_\alpha \longrightarrow N_\alpha \longrightarrow N''_\alpha \longrightarrow 0. \qquad (*)$$

Now, $N' \cap N_\alpha = \varinjlim_\beta \mathcal{N}_\beta^{(\alpha)}$, where $\mathcal{N}_\beta^{(\alpha)}$ ranges over the f.g. submodules of $N' \cap N_\alpha$. We get the exact sequence

$$0 \longrightarrow \mathcal{N}_\beta^{(\alpha)} \longrightarrow N_\alpha \longrightarrow N''_{\alpha,\beta} \longrightarrow 0, \tag{†}$$

where $N''_{\alpha,\beta} = N_\alpha/\mathcal{N}_\beta^{(\alpha)}$, and all the modules in (†) are f.g. The right limit of (†) is (∗). By hypothesis, $M \otimes_\Omega$ (†) is still exact, and the right limit of an exact sequence is exact; so

$$0 \longrightarrow M \otimes_\Omega (N' \cap N_\alpha) \longrightarrow M \otimes_\Omega N_\alpha \longrightarrow M \otimes_\Omega N''_\alpha \longrightarrow 0 \quad \text{is exact.}$$

Now, if we pass to the right limit, this time over α, we get

$$0 \longrightarrow M \otimes_\Omega N' \longrightarrow M \otimes_\Omega N \longrightarrow M \otimes_\Omega N'' \longrightarrow 0 \quad \text{is exact.}$$ ☐

Theorem 2.68 *(FGI-Test)*[1] *An Ω-module, M, is flat iff for all sequences*

$$0 \longrightarrow \mathfrak{A} \longrightarrow \Omega^{\mathrm{op}} \longrightarrow \Omega^{\mathrm{op}}/\mathfrak{A} \longrightarrow 0$$

in which $\mathfrak{A}$ is a finitely generated Ω^{op}-ideal, the sequence

$$0 \longrightarrow \mathfrak{A} \otimes_\Omega M \longrightarrow \Omega^{\mathrm{op}} \otimes_\Omega M \longrightarrow (\Omega^{\mathrm{op}}/\mathfrak{A}) \otimes_\Omega M \longrightarrow 0 \quad \textit{is still exact.}$$

Proof. (⇒) is trivial.

(⇐). We proceed in two steps.

Step 1. I claim: For every exact sequence of Ω^{op}-modules of the form

$$0 \longrightarrow K \longrightarrow \coprod_S \Omega^{\mathrm{op}} \longrightarrow N \longrightarrow 0, \tag{∗}$$

in which $\#(S)$ is finite, we have an exact sequence

$$0 \longrightarrow K \otimes_\Omega M \longrightarrow \Big(\coprod_S \Omega^{\mathrm{op}}\Big) \otimes_\Omega M \longrightarrow N \otimes_\Omega M \longrightarrow 0.$$

We prove this by induction on the *minimal number, r, of generators of N*. (Note that $\#(S) \geq r$.) The case $r = 1$ has all the ingredients of the general proof as we will see. When $r = 1$, look first at the base case: $\#(S) = 1$, too. Sequence (∗) is then:

$$0 \longrightarrow K \longrightarrow \Omega^{\mathrm{op}} \longrightarrow N \longrightarrow 0. \tag{$(*)_1$}$$

This means that K is an ideal of Ω^{op} and we know $K = \varinjlim_\alpha K_\alpha$, where the K_α's are f.g. Ω^{op}-ideals. Then, $(*)_1$ is the right limit of

$$0 \longrightarrow K_\alpha \longrightarrow \Omega^{\mathrm{op}} \longrightarrow N_\alpha \longrightarrow 0, \tag{$(*)_\alpha$}$$

where $N_\alpha = \Omega^{\mathrm{op}}/K_\alpha$. Our hypothesis shows that

$$0 \longrightarrow K_\alpha \otimes_\Omega M \longrightarrow \Omega^{\mathrm{op}} \otimes_\Omega M \longrightarrow N_\alpha \otimes_\Omega M \longrightarrow 0 \quad \text{is exact.}$$

Pass the latter sequence to the limit over α and obtain

$$0 \longrightarrow K \otimes_\Omega M \longrightarrow \Omega^{\mathrm{op}} \otimes_\Omega M \longrightarrow N \otimes_\Omega M \longrightarrow 0 \quad \text{is exact.}$$

[1]FGI stands for finitely generated ideal.

Thus, the base case $\#(S) = r = 1$ is proved.

We now use induction on $\#(S)$ to establish the case $\#(S) > r = 1$. (So, our claim involves an induction inside an induction.) The induction hypothesis is: For all exact sequences

$$0 \longrightarrow K \longrightarrow \coprod_S \Omega^{\mathrm{op}} \longrightarrow N \longrightarrow 0,$$

in which $\#(S) = s$ *and* r (= minimal number of generators of N) = 1, tensoring with M leaves the sequence exact. Say it is true for all sequences with $\#(S) < s$. Given

$$0 \longrightarrow K \longrightarrow \coprod_S \Omega^{\mathrm{op}} \longrightarrow N \longrightarrow 0, \quad \#(S) = s,$$

pick some $\sigma \in S$ and let $\Sigma = S - \{\sigma\}$. We have the map $\Omega^{\mathrm{op}} = \Omega_\sigma^{\mathrm{op}} \hookrightarrow \coprod_S \Omega^{\mathrm{op}} \longrightarrow N$, and we let N_σ be the image of this map in N. This gives the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_\sigma & \longrightarrow & \Omega = \Omega_\sigma & \longrightarrow & N_\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \coprod_S \Omega & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'' & \longrightarrow & \coprod_\Sigma \Omega & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(where $N'' = N/N_\sigma$) with exact rows and columns and the middle column split-exact. Note that N'' and N_σ have $r \leq 1$ and when $r = 0$ the above argument is trivial. Tensor the diagram on the right with M. So, the top and bottom rows remain exact (by the induction hypothesis and the base case), the middle column remains exact (in fact, split) and all other rows and columns are exact:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & K_\sigma \otimes_\Omega M & \longrightarrow & \Omega \otimes_\Omega M & \longrightarrow & N_\alpha \otimes_\Omega M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu} & & \downarrow & & \downarrow & & \\
 & & K \otimes_\Omega M & \xrightarrow{\alpha} & \left(\coprod_S \Omega\right) \otimes_\Omega M & \longrightarrow & N \otimes_\Omega M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\
0 & \longrightarrow & K'' \otimes_\Omega M & \longrightarrow & \left(\coprod_\Sigma \Omega\right) \otimes_\Omega M & \longrightarrow & N'' \otimes_\Omega M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

We must show that α is an injection. Take $x \in K \otimes_\Omega M$. If $\alpha(x) = 0$, then $\theta(\alpha(x)) = 0$, which implies that $\pi(x)$ goes to zero under the injection $(K'' \otimes_\Omega M \longrightarrow \left(\coprod_\Sigma \Omega\right) \otimes_\Omega M)$, and so, $\pi(x) = 0$. Then, there is some

$y \in K_\sigma \otimes_\Omega M$ with $\nu(y) = x$. But the map $K_\sigma \otimes_\Omega M \longrightarrow \Omega \otimes_\Omega M \longrightarrow \left(\coprod_S \Omega\right) \otimes_\Omega M$ is injective and y goes to zero under it. So, we must have $y = 0$, and thus, $x = 0$. This proves that α is injective, and completes the interior induction (case: $r = 1$). By the way, α is injective by the five lemma with the two left vertical sequences considered horizontal and read backwards!

There remains the induction on r. The case $r = 1$ is proved. If the statement is true for modules N with $< r$ minimal generators, we take an N with exactly r as its number of minimal generators. Then, for any finite S, and any sequence

$$0 \longrightarrow K \longrightarrow \coprod_S \Omega^{\mathrm{op}} \longrightarrow N \longrightarrow 0,$$

we choose, as above, $\sigma \in S$ and set $\Sigma = S - \{\sigma\}$ and let N_σ, N'' be as before. Now redo the argument involving the 9 term diagram; it shows α is, once again, injective and the claim is proved.

Step 2. I claim that for every sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of Ω^{op}-modules, all of which are f.g., the sequence

$$0 \longrightarrow N' \otimes_\Omega M \longrightarrow N \otimes_\Omega M \longrightarrow N'' \otimes_\Omega M \longrightarrow 0$$

remains exact. By the previous proposition, this will finish the proof.

Since N', N and N'' are all f.g., we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & \coprod_S \Omega & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \coprod_{S\cup T} \Omega & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'' & \longrightarrow & \coprod_T \Omega & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

in which the middle column is split-exact. By tensoring this diagram with M (on the right), we get the following commutative diagram with all exact rows (by Step 1) and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & \mathrm{Ker}\ \alpha & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' \otimes_\Omega M & \longrightarrow & \left(\coprod_S \Omega\right) \otimes_\Omega M & \longrightarrow & N' \otimes_\Omega M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & K \otimes_\Omega M & \longrightarrow & \left(\coprod_{S\cup T} \Omega\right) \otimes_\Omega M & \longrightarrow & N \otimes_\Omega M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'' \otimes_\Omega M & \stackrel{\beta}{\longrightarrow} & \left(\coprod_T \Omega\right) \otimes_\Omega M & \longrightarrow & N'' \otimes_\Omega M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}
$$

We must show that α is injective. Apply the snake lemma to the first two rows: We get

$$0 \longrightarrow \mathrm{Ker}\ \alpha \stackrel{\delta}{\longrightarrow} K'' \otimes_\Omega M \stackrel{\beta}{\longrightarrow} \Big(\coprod_T \Omega\Big) \otimes_\Omega M \quad \text{is exact.}$$

But, $\mathrm{Ker}\ \beta = (0)$ implies that $\mathrm{Im}\ \delta = (0)$, and so, $\mathrm{Ker}\ \alpha = (0)$. $\square$

The second (unproven) assertion of Proposition 2.54 now follows from Theorem 2.68.

Corollary 2.69 *If Ω is a P.I.D., more generally, a nonzero-divisor ring all of whose f.g. Ω^{op}-ideals are principal, then M is flat over Ω iff M is Ω-torsion-free.*

Proof. The implication $(\Rightarrow)$ is always true when Ω has no zero divisors.

$(\Leftarrow)$. By the previous theorem, we only need to test against exact sequences of the form

$$0 \longrightarrow \mathfrak{A} \longrightarrow \Omega^{\mathrm{op}} \longrightarrow \Omega^{\mathrm{op}}/\mathfrak{A} \longrightarrow 0,$$

where $\mathfrak{A}$ is a f.g. (hence, *principal*) Ω^{op}-ideal. So, there is some $\lambda \in \Omega$ with $\mathfrak{A} = \lambda\Omega$. We have the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega & \stackrel{\lambda}{\longrightarrow} & \Omega & \longrightarrow & \Omega/\lambda\Omega & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta} & & \| & & \| & & \\
0 & \longrightarrow & \mathfrak{A} & \hookrightarrow & \Omega & \longrightarrow & \Omega/\mathfrak{A} & \longrightarrow & 0
\end{array}
$$

(with $\mathfrak{A}$ considered as right ideal and where $\theta(\mu) = \lambda\mu$) and all the vertical maps are isomorphisms. Consequently, we may assume that our exact sequence is

$$0 \longrightarrow \Omega \stackrel{\lambda}{\longrightarrow} \Omega \longrightarrow \Omega/\lambda\Omega \longrightarrow 0.$$

By tensoring with M, we get the exact sequence

$$\Omega \otimes_\Omega M \stackrel{\lambda}{\longrightarrow} \Omega \otimes_\Omega M \longrightarrow (\Omega/\lambda\Omega) \otimes_\Omega M \longrightarrow 0,$$

which, in view of the isomorphisms $\Omega \otimes_\Omega M \cong M$ and $(\Omega/\lambda\Omega) \otimes_\Omega M \cong M/\lambda M$, is equivalent to

$$M \stackrel{\lambda}{\longrightarrow} M \longrightarrow M/\lambda M \longrightarrow 0.$$

Since M has no torsion, multiplication by λ is injective and the sequence is exact. $\square$

The corollary is false if Ω is not a P.I.D. Here is an example:

Consider the ring, $A = \mathbb{C}[X, Y]$ ($A \in \mathrm{CR}$). The ring A is a domain; so, it is torsion-free. (It's even a UFD.) Let $\mathfrak{M}$ be the ideal of A generated by X and Y. We can write

$$\begin{aligned}\mathfrak{M} &= \{f \in \mathbb{C}[X, Y] \mid f(X, Y) = g(X, Y)X + h(X, Y)Y, \text{ with } g(X, Y), h(X, Y) \in \mathbb{C}[X, Y]\}\\ &= \{f \in \mathbb{C}[X, Y] \mid f(0, 0) = 0, \text{ i.e., } f \text{ has no constant term}\}.\end{aligned}$$

Since $\mathfrak{M} \subseteq A$, we see that $\mathfrak{M}$ is torsion-free.

Claim: $\mathfrak{M}$ is not flat.

Now, $A/\mathfrak{M} \cong \mathbb{C}$, so $\mathbb{C}$ is an A-module; how?

The A-module structure on $\mathbb{C}$ is as follows: For any $f(X, Y) \in A$ and any $\lambda \in \mathbb{C}$,

$$f(X, Y) \cdot \lambda = f(0, 0)\lambda.$$

Note that $X \cdot \lambda = Y \cdot \lambda = 0$. When we consider $\mathfrak{M}$ as an A-module, write its generators as e_1 and e_2. Under the map $\mathfrak{M} \longrightarrow A$, we have $e_1 \mapsto X$ and $e_2 \mapsto Y$. There is a unique nontrivial relation:

$$Y \cdot e_1 - X \cdot e_2 = 0.$$

We claim that $e_1 \otimes e_2 \neq e_2 \otimes e_1$ in $\mathfrak{M} \otimes_A \mathfrak{M}$. To see this, define a map, $B \colon \mathfrak{M} \times \mathfrak{M} \to \mathbb{C}$.

(a) First, define B on the generators e_1, e_2, by setting

$$B(e_1, e_1) = B(e_2, e_2) = 0, \quad B(e_1, e_2) = 1, \quad B(e_2, e_1) = -1.$$

(b) We need to check that B is well-defined. Let's check it for the left hand side argument:

$$B\left(Y \cdot e_1 - X \cdot e_2, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right) = Y \cdot B\left(e_1, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right) - X \cdot B\left(e_2, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right).$$

In the case of e_1, we get $X \cdot 1 = 0$, and in the case of e_2, we get $Y \cdot 1 = 0$. The reader should check similarly that there is no problem for the righthand side argument.

Consequently, we get a linear map, $\theta \colon \mathfrak{M} \otimes \mathfrak{M} \longrightarrow \mathbb{C}$. For this linear map,

$$\theta(e_1 \otimes e_1) = \theta(e_2 \otimes e_2) = 0, \quad \theta(e_1 \otimes e_2) = 1, \quad \theta(e_2 \otimes e_1) = -1.$$

So, $e_1 \otimes e_2 \neq e_2 \otimes e_1$, as contended. Now we will see that $\mathfrak{M}$ is **not** flat as A-module. Look at the exact sequence

$$0 \longrightarrow \mathfrak{M} \longrightarrow A \longrightarrow \mathbb{C} \longrightarrow 0$$

and tensor it with $\mathfrak{M}$. We get

$$\mathfrak{M} \otimes_A \mathfrak{M} \longrightarrow A \otimes_A \mathfrak{M} \longrightarrow \mathbb{C} \otimes_A \mathfrak{M} \longrightarrow 0 \quad \text{is exact.}$$

However, $\mathfrak{M} \otimes_A \mathfrak{M} \longrightarrow A \otimes_A \mathfrak{M}$ is not injective. To see this, use the isomorphism $\mu \colon A \otimes_A \mathfrak{M} \cong \mathfrak{M}$, *via* $\alpha \otimes m \mapsto \alpha \cdot m$ and examine the composed homomorphism

$$\varphi \colon \mathfrak{M} \otimes_A \mathfrak{M} \longrightarrow A \otimes_A \mathfrak{M} \xrightarrow{\mu} \mathfrak{M}.$$

Since μ is an isomorphism, all we must prove is that φ is not injective. But,

$$\begin{aligned}\varphi(e_1 \otimes e_2) &= \mu(X \otimes e_2) = X \cdot e_2\\ \varphi(e_2 \otimes e_1) &= \mu(Y \otimes e_1) = Y \cdot e_1.\end{aligned}$$

Yet, $X \cdot e_2 = Y \cdot e_1$ and $e_1 \otimes e_2 \neq e_2 \otimes e_1$, so φ is not injective and $\mathfrak{M}$ is not flat.

Say Ω is a Λ-algebra and M is a Λ^{op}-module, then $M \otimes_\Lambda \Omega$ is an Ω^{op}-module. The module $M \otimes_\Lambda \Omega$ is called the *base extension* of M to Ω.

Proposition 2.70 *Say M is a flat Λ-module, then its base extension, $\Omega \otimes_\Lambda M$, is again a flat Ω-module. If N is a flat Ω-module and Ω is a flat Λ-algebra, then N considered as Λ-module (via $\Lambda \longrightarrow \Omega$), is again flat over Λ.*

Proof. Assume M is flat as Λ-module. Then, we know that for any exact sequence of Λ^{op}-modules,

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

the sequence

$$0 \longrightarrow N' \otimes_\Lambda M \longrightarrow N \otimes_\Lambda M \quad \text{is exact.}$$

Now, take any exact sequence of Ω-modules, say

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0, \tag{$\dagger$}$$

it is still exact as a sequence of Λ-modules. Hence,

$$0 \longrightarrow N' \otimes_\Lambda M \longrightarrow N \otimes_\Lambda M \quad \text{is exact.}$$

Tensoring ($\dagger$) with $\Omega \otimes_\Lambda M$ over Ω, we get

$$N' \otimes_\Omega (\Omega \otimes_\Lambda M) \longrightarrow N \otimes_\Omega (\Omega \otimes_\Lambda M) \longrightarrow \cdots . \tag{$\dagger\dagger$}$$

We want to show that ($\dagger\dagger$) is exact on the left. But $Z \otimes_\Omega (\Omega \otimes_\Lambda M) \cong Z \otimes_\Lambda M$, for any Ω^{op}-module, Z. Hence, ($\dagger\dagger$) becomes

$$N' \otimes_\Lambda M \longrightarrow N \otimes_\Lambda M \longrightarrow \cdots ,$$

and we already observed that this sequence is exact on the left.

For the second part, take an exact sequence of Λ^{op}-modules,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{$*$}$$

We need to show that

$$0 \longrightarrow M' \otimes_\Lambda N \longrightarrow M \otimes_\Lambda N \quad \text{is exact.}$$

Tensor ($*$) over Λ with Ω. The resulting sequence

$$0 \longrightarrow M' \otimes_\Lambda \Omega \longrightarrow M \otimes_\Lambda \Omega \longrightarrow \cdots \tag{$**$}$$

is still exact as Ω is flat. Tensor ($**$) with N over Ω; again, as N is flat over Ω, we get

$$0 \longrightarrow (M' \otimes_\Lambda \Omega) \otimes_\Omega N \longrightarrow (M \otimes_\Lambda \Omega) \otimes_\Omega N \longrightarrow \cdots \quad \text{is exact.}$$

But the latter exact sequence is just

$$0 \longrightarrow M' \otimes_\Lambda N \longrightarrow M \otimes_\Lambda N \longrightarrow \cdots ,$$

as required. □

Harder question: Let $P(\Lambda)$ be a property of Λ-modules. Say Ω is a Λ-algebra and M is a Λ-module. Then, we get the Ω-module, $\Omega \otimes_\Lambda M$, the base extension of M to Ω. Suppose, $\Omega \otimes_\Lambda M$ has $P(\Omega)$. Does M have $P(\Lambda)$?

If so, one says that *P descends in the extension Ω over Λ*. This matter is a question of descent.

A more realistic question is: Given P, or a collection of interesting P's, for which Λ-algebras, Ω, does (do) $P(\Omega)$ descend?

Examples:

1. $P_1(\Lambda)$: M is a torsion-free Λ-module.
2. $P_2(\Lambda)$: M is a flat Λ-module.
3. $P_3(\Lambda)$: M is a free Λ-module.
4. $P_4(\Lambda)$: M is an injective Λ-module.
5. $P_5(\Lambda)$: M is a torsion Λ-module.

Take $\Lambda = \mathbb{Z}$ (a very good ring: commutative, P.I.D), $\Omega = \mathbb{Q}$ (a field, a great ring), $\mathbb{Q}$ is flat over $\mathbb{Z}$ (and $\mathbb{Z} \hookrightarrow \mathbb{Q}$). Let $M = \mathbb{Z} \amalg (\mathbb{Z}/2\mathbb{Z})$. (The module M is f.p.) The module M, has, **none** of $P_j(\mathbb{Z})$ for $j = 1, 2, 3, 4$. On the other hand, $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q}$, and $\mathbb{Q}$ has all of $P_j(\mathbb{Q})$ for $j = 1, 2, 3, 4$. However, P_5 descends in the extension $\mathbb{Q}$ over $\mathbb{Z}$. This follows from

Proposition 2.71 *The module, M, is a torsion $\mathbb{Z}$-module iff $\mathbb{Q} \otimes_{\mathbb{Z}} M = (0)$.*

Proof. $(\Rightarrow)$. This has already been proved.

$(\Leftarrow)$. First, let M be f.g. We know that there is an exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0 \qquad (\dagger)$$

where $t(M)$ is the torsion submodule of M and $M/t(M)$ is torsion-free; hence (since M is f.g.), $M/t(M)$ is free. If we tensor $(\dagger)$ with $\mathbb{Q}$, we get

$$\mathbb{Q} \otimes_{\mathbb{Z}} M \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (M/t(M)) \longrightarrow 0.$$

Since $\mathbb{Q} \otimes_{\mathbb{Z}} M = (0)$, by hypothesis, we get $\mathbb{Q} \otimes_{\mathbb{Z}} (M/t(M)) = (0)$. Yet, $M/t(M) = \amalg_S \mathbb{Z}$ where S is finite; consequently, $S = \emptyset$ and so, $M/t(M) = (0)$, i.e., $M = t(M)$. Therefore, M is torsion.

For an arbitrary M, we can write $M = \varinjlim_{\alpha} M_\alpha$, where M_α ranges over the f.g. submodules of M. We have an exact sequence

$$0 \longrightarrow M_\alpha \longrightarrow M, \quad \text{for all } \alpha,$$

and $\mathbb{Q}$ is flat; so,

$$0 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} M_\alpha \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} M \quad \text{is still exact.}$$

But, $\mathbb{Q} \otimes_{\mathbb{Z}} M = (0)$ implies $\mathbb{Q} \otimes_{\mathbb{Z}} M_\alpha = (0)$. As the M_α's are f.g., the previous argument shows that M_α is torsion. Then, $M = \varinjlim_{\alpha} M_\alpha$ is torsion as the right limit of torsion modules is torsion. ☐

We now go back to the question. Given the $\mathbb{Z}$-module M, we assume that $\mathbb{Q} \otimes_{\mathbb{Z}} M$ is torsion. Since $\mathbb{Q}$ is a field, $\mathbb{Q} \otimes_{\mathbb{Z}} M = (0)$. Proposition 2.71 implies that M is torsion and P_5 descends in the extension $\mathbb{Q}$ over $\mathbb{Z}$.

2.9 Further Readings

Rings and modules are covered in most algebra texts, so we shall nor repeat the references given in Section 1.8. Other references include Atiyah MacDonald [3], Lafon [32, 33], Eisenbud [13], Matsumura [39], Malliavin [38] and Bourbaki [8].