1. Let $D_8$ be the dihedral group with 8 elements, with generators $a, b$ such that $a^4 = 1 = b^2$, $bab^{-1} = a^{-1}$. Let $H$ be the subgroup of $D_8$ generated by $a^2$ and $b$.

   (i) Show that the 1-dimensional subspace $V := \mathbb{C} \cdot (1 - a^2 + b - a^2b)$ of $\mathbb{C}[H]$ is an ideal in $\mathbb{C}[H]$.

   (ii) Write down the character of $V$.

   (iii) Write down the character of the induced representation $\text{Ind}_{H}^{D_8}(V)$. Is it irreducible?

2. Let $G$ be a finite group, $H$ be a subgroup of $G$, and let $k$ be a field. Let $V$ be a left $k[H]$-module. Define $W$ to be the $k$-vector space of all functions $f : G \rightarrow V$ such that $f(hx) = h \cdot f(x)$ for all $h \in H$ and all $x \in G$.

   (i) For every element $y \in G$ and every element $f \in W$, define $y \cdot f$ to be the element in $W$ such that $(y \cdot f)(x) = f(xy)$ for every $x \in G$. Show that this gives $W$ a structure as a left $k[G]$-module.

   (ii) Show that $W$ is isomorphic to $k[G] \otimes_{k[H]} V$ as a $k[G]$-module.

3. Let $G$ be a finite group and let $k$ be a field. Let $V$ be a finite dimensional $k$-linear representation of $G$. Consider the tensor product $k[G] \otimes_k V$, the tensor product of the left regular representation with $V$, and regard $k[G] \otimes_k V$ as a left $k[G]$-module. Show that $k[G] \otimes_k V$ is a free left $k[G]$-module.

4. Let $G$ be a finite group, and let $H$ be a subgroup of $G$.

   (i) Let $\chi$ be an irreducible complex character of $G$. Write $\text{Res}_{H}^{G}(\chi) = \sum_{j=1}^{a} d_j \omega_j$, where the $\omega_j$’s are distinct irreducible complex characters of $H$. Show that $\sum_{j=1}^{a} d_j^2 \leq [G : H]$.

   (ii) Let $\omega$ be an irreducible complex character of $H$. Write $\text{Ind}_{H}^{G}(\omega) = \sum_{i=1}^{b} e_i \chi_i$, where the $\chi_i$’s are irreducible complex characters of $G$. Show that $\sum_{i=1}^{b} e_i^2 \leq [G : H]$.

5. Let $G$ be a finite group, and let $k$ be a field.

   (i) Show that the free left $k[G]$-module is an injective $k[G]$-module. (Hint: Use the construction $V \sim V^V$ for finite-dimensional $k$-linear representations of $G$. Also, See Shatz-Gallier for the definition of injective modules.)

   (ii) Show that every projective left $k[G]$-module is injective.

6. Let $G$ be a finite group and let $\rho : G \rightarrow \text{GL}(V)$ be an irreducible linear representation of $G$ on a finite dimensional vector space $V$ over $\mathbb{C}$. Let $N \subseteq G$ be the subgroup of $G$ consisting of all elements $x \in G$ such that $\rho(x) \in \mathbb{C}^\times \text{Id}_V$.

   (i) Let $G^n = G \times \cdots \times G$ be the product of $n$ copies of $G$, $n \geq 1$. $\rho_n : G^n \rightarrow V \otimes \cdots \otimes V$ be the $n$-th exterior tensor power of $(\rho, V)$. Let $N_n$ be the subgroup of $N_n := N \times \cdots \times N$ consisting of all elements $(x_1, \ldots, x_n) \in N^n$ such that $x_1 \cdot \cdots \cdot x_n = 1$. Show that $\dim_{\mathbb{C}}(V)^n$ divides $\text{Card}(G^n/N_n)$.

   (ii) Show that $\dim_{\mathbb{C}}(V)$ divides $\text{Card}(G/N)$.
7. Let \( \rho : G \to GL(V) \) be a faithful finite dimensional complex representation of a finite group \( G \). Let \( \chi = \chi_\rho \) be the character of \( \rho \). Let \( \omega \) be an irreducible complex character of \( G \). For every \( m \in \mathbb{N} \), denote by \( \chi_m \) the character of \( V^\otimes m \).

(i) Let \( a_m := \langle \chi_m, \omega \rangle \) be the multiplicity of \( \omega \) in \( V^\otimes m \). Show that the generating function

\[
f(t) := \sum_{m \geq 0} a_m t^m \in \mathbb{C}[[t]]
\]

is a rational function and give a formula for the rational function \( f(t) \) in terms of the characters \( \chi \) and \( \omega \).

(ii) Determine the radius \( r_0 \) of convergence of the power series \( f(t) \).

(iii) Determine the order of poles of \( f(t) \) at points of the circle \( |t| = r_0 \).

(iv) Show that the number of poles of \( f(t) \) at points of the circle \( |t| = r_0 \) is equal to the number of conjugacy classes of \( G \) contained in the normal subgroup \( \rho^{-1}(\mathbb{C}^\times \text{Id}_V) \).

(v) Show that \( f(t) \) is not identically equal to 0. Deduce that \( \chi \) appears in \( V^\otimes n \) for some \( n \in \mathbb{N} \).

(vi) Show that there exists infinitely many natural numbers \( n \in \mathbb{N} \) such that \( \chi \) appears in \( V^\otimes n \).