Math 602 Assignment 11, Fall 2006

1. Let D_8 be the dihedral group with 8 elements, with generators a, b such that $a^4 = 1 = b^2$, $bab^{-1} = a^{-1}$. Let H be the subgroup of D_8 generated by a^2 and b.

(i) Show that the 1-dimensional subspace $V := \mathbb{C} \cdot (1 - a^2 + b - a^2 b)$ of $\mathbb{C}[H]$ is an ideal in $\mathbb{C}[H]$.

(ii) Write down the character of V.

(iii) Write down the character of the induced representation $\mathrm{Ind}_H^{D_8}(V)$. Is it irreducible?

2. Let G be a finite group, H be a subgroup of G, and let k be a field. Let V be a left $k[G]$-module. Define W to be the k-vector space of all functions $f : G \to V$ such that $f(hx) = h \cdot f(x)$ for all $h \in H$ and all $x \in G$.

(i) For every element $y \in G$ and every element $f \in W$, define $y \cdot f$ to be the element in W such that $(y \cdot f)(x) = f(xy)$ for every $x \in G$. Show that this gives W a structure as a left $k[G]$-module.

(ii) Show that W is isomorphic to $k[G] \otimes_{k[H]} V$ as a $k[G]$-module.

3. Let G be a finite group and let k be a field. Let V be a finite dimensional k-linear representation of G. Consider the tensor product $k[G] \otimes_k V$, the tensor product of the left regular representation with V, and regard $k[G] \otimes_k V$ as a left $k[G]$-module. Show that $k[G] \otimes_k V$ is a free left $k[G]$-module.

4. Let G be a finite group, and let H be a subgroup of G.

(i) Let χ be an irreducible complex character of G. Write $\mathrm{Res}_H^G(\chi) = \sum_{j=1}^a d_j \, \omega_j$, where the ω_j's are distinct irreducible complex characters of H. Show that $\sum_{j=1}^a d_j^2 \leq [G : H]$.

(ii) Let ω be an irreducible complex character of H. Write $\mathrm{Ind}_H^G(\omega) = \sum_{i=1}^b e_i \, \chi_i$, where the χ_i's are irreducible complex characters of G. Show that $\sum_{i=1}^b e_i^2 \leq [G : H]$.

5. Let G be a finite group, and let k be a field.

(i) Show that the free left $k[G]$-module is an injective $k[G]$-module. (Hint: Use the construction $V \rightsquigarrow V^\vee$ for finite-dimensional k-linear representations of G. Also, See Shatz-Gallier for the definition of injective modules.)

(ii) Show that every projective left $k[G]$-module is injective.

6. Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be an irreducible linear representation of G on a finite dimensional vector space V over $\mathbb{C}$. Let $N \subseteq G$ be the subgroup of G consisting of all elements $x \in G$ such that $\rho(x) \in \mathbb{C}^\times \mathrm{Id}_V$.

(i) Let $G^n = G \times \cdots \times G$ be the product of n copies of G, $n \geq 1$. $\rho_n : G^n \to V \otimes \cdots \otimes V$ be the n-th exterior tensor power of (ρ, V). Let N_n be the subgroup of $N_n := N \times \cdots \times N$ consisting of all elements $(x_1, \ldots, x_n) \in N^n$ such that $x_1 \cdot \cdots \cdot x_n = 1$. Show that $\dim_{\mathbb{C}}(V)^n$ divides $\mathrm{Card}(G^n / N_n)$.

(ii) Show that $\dim_{\mathbb{C}}(V)$ divides $\mathrm{Card}(G/N)$.

7. Let $\rho : G \to GL(V)$ be a faithful finite dimensional complex representation of a finite group G. Let $\chi = \chi_\rho$ be the character of ρ. Let ω be an irreducible complex character of G. For every $m \in \mathbb{N}$, denote by χ_m the character of $V^{\otimes m}$.

(i) Let $a_m := \langle \chi_m, \omega \rangle$ be the multiplicity of ω in $V^{\otimes m}$. Show that the generating function

$$f(t) := \sum_{m \geq 0} a_m t^n \in \mathbb{C}[[t]]$$

is a rational function and give a formula for the rational function $f(t)$ in terms of the characters χ and ω.

(ii) Determine the radius r_0 of convergence of the power series $f(t)$.

(iii) Determine the order of poles of $f(t)$ at points of the circle $|t| = r_0$.

(iv) Show that the number of poles of $f(t)$ at points of the circle $|t| = r_0$ is equal to the number of conjugacy classes of G contained in the normal subgroup $\rho^{-1}(\mathbb{C}^\times \operatorname{Id}_V)$.

(v) Show that $f(t)$ is not identically equal to 0. Deduce that χ appears in $V^{\otimes n}$ for some $n \in \mathbb{N}$.

(vi) Show that there exists infinitely many natural numbers $n \in \mathbb{N}$ such that χ appears in $V^{\otimes n}$.