Let \( G \) be the generator of \( \mathbb{Z} \) the element \( \mathbb{Z} \) semi-direct product \((\mathbb{Z}/2\mathbb{Z}) \rtimes_\rho (\mathbb{Z}/2\mathbb{Z})\) for the action \( \rho : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/2^{n-1}\mathbb{Z})\), where the generator of \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z}/2^{n-1}\mathbb{Z} \) as \( a \mapsto -a \forall a \in \mathbb{Z}/2^{n-1}\mathbb{Z} \).

(ii) Let \( n \geq 4 \) be a positive integer. Let \( \text{SD}_{2^n} \) be the semi-direct product \((\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/2\mathbb{Z})\) for the action \( \tau : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/2^{n-1}\mathbb{Z})\), where the generator of \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z}/2^{n-1}\mathbb{Z} \) as \( a \mapsto (2^{n-2} - 1)a \forall a \in \mathbb{Z}/2^{n-1}\mathbb{Z} \).

(iii) Let \( n \geq 3 \) be a positive integer. Let \( G \) be the semi-direct product \((\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_\mu (\mathbb{Z}/4\mathbb{Z})\) for the action \( \mu : \mathbb{Z}/4\mathbb{Z} \to \text{Aut}(\mathbb{Z}/2^{n-1}\mathbb{Z})\), where the generator \( 1 = 1 + 4\mathbb{Z} \) of \( \mathbb{Z}/4\mathbb{Z} \) operates on \( \mathbb{Z}/2^{n-1}\mathbb{Z} \) as \( a \mapsto -a \forall a \in \mathbb{Z}/2^{n-1}\mathbb{Z} \). Notice that \( \mathbb{Z}(G) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^2\), generated by the elements \((2^{n-2} + 2^{n-1}Z, 4Z), (2^{n-1}Z, 2 + 4Z)\). Define the quaternion group \( Q_{2^n} \) with \( 2^n \) elements to be the quotient of \( G \) by the subgroup of \( \mathbb{Z}(G) \) generated by the element \((2 + 2^{n-1}Z, 2 + 4Z)\) of order two.

1. For \( G = \text{Mod}_{p^n} \) determine the Frattini subgroup \( \Phi(G) \), the commutator subgroup \([G, G]\), the \( p \)-rank of \( G \) (i.e. the largest integer \( m \) such that \( G \) contains a subgroup isomorphic to \( \mathbb{Z}^m \)), the ascending central series of \( G \) and the descending central series of \( G \).

2. For \( G = \text{D}_{2^n}, \text{SD}_{2^n}, \text{Q}_{2^n} \), determine the Frattini subgroup \( \Phi(G) \), the commutator subgroup \([G, G]\), the \( 2 \)-rank of \( G \), the ascending central series of \( G \) and the descending central series of \( G \).

3. Suppose that \( G \) is a non-commutative (finite) \( p \)-group for a prime number \( p \) such that \( G \) has a cyclic normal subgroup of index \( p \). Prove that \( G \) is isomorphic to a \( \text{Mod}_{p^n} \) if \( p \) is odd, and \( G \) is isomorphic to a \( \text{Mod}_{2^n}, \text{D}_{2^n}, \text{SD}_{2^n} \) or \( \text{Q}_{2^n} \) if \( p = 2 \).

4. Let \( G \) be a finite \( p \)-group, where \( p \) is a prime number. Show that there is a characteristic subgroup \( H \) of \( G \) such that \( \Phi(H) \leq Z(H) = Z_G(H) \geq [G, H] \). Here \([G, H]\) is the subgroup of \( G \) generated by commutators of the form \( x^{-1}y^{-1}xy \) with \( x \in G, y \in H \). A subgroup of \( G \) with the above properties is called a critical subgroup of \( G \). (Hint: Consider the partially order set \( \mathcal{S} \) of all characteristic subgroups \( H \) of \( G \) such that \( \Phi(H) \leq Z(H) \geq [G, H] \), and show that every maximal element in \( \mathcal{S} \) is a critical subgroup of \( G \).)

5. Find a critical subgroup for each of the following groups: \( \text{Mod}_{p^n}, \text{D}_{2^n}, \text{SD}_{2^n}, \text{Q}_{2^n} \).

6. Let \( G = S_3 = \text{D}_3 \), and let \( N \) be the normal subgroup with 3 elements in \( G \). Denote by \( e_N \) the element
\[
e_N := \frac{1}{3} \sum_{x \in N} x \in \mathbb{Q}[G]
\]
in the rational group algebra \( \mathbb{Q}[G] \) of \( G \).

(i) Show that \( e_N \in Z(\mathbb{Q}[G]) \), and \( e_N^2 = e_N \). Consequently the ideals \( e_N \mathbb{Q}[G] = e_N \mathbb{Q}[G]e_N = \mathbb{Q}[G]e_N \) and \((1 - e_N)\mathbb{Q}[G] = (1 - e_N)\mathbb{Q}[G](1 - e_N) = \mathbb{Q}[G](1 - e_N)\) have natural structure as rings with unit, and we have a natural isomorphism
\[
\mathbb{Q}[G] = e_N \mathbb{Q}[G] \times (1 - e_N)\mathbb{Q}[G](1 - e_N)
\]
of \( \mathbb{Q} \)-algebras.
(ii) Prove that the map \( x \to e_N x \) defines a surjective ring homomorphism \( \pi \) from \( \mathbb{Q}[G] \) to \( \mathbb{Q}[G/N] \cong \mathbb{Q}[\mathbb{Z}/2\mathbb{Z}] \) whose kernel is equal to \( (1 - e_N)\mathbb{Q}[G](1 - e_N) \). Show that \( \mathbb{Q}[G/N] \) is isomorphic to \( \mathbb{Q} \times \mathbb{Q} \) as \( \mathbb{Q} \)-algebras.

(iii) Show that \( \mathbb{Q}[G]/e_N \mathbb{Q}[G]e_N \cong (1 - e_N)\mathbb{Q}[G](1 - e_N) \) is a four-dimensional simple \( \mathbb{Q} \)-algebra, i.e. it has no non-trivial two-sided ideals.

(iv) Is \( \mathbb{Q}[G]/e_N \mathbb{Q}[G]e_N \) isomorphic to \( M_2(\mathbb{Q}) \)? (Either establish an isomorphism or show that no such isomorphism exists.)

7. Let \( p \geq 5 \) be prime number. Let \( G = D_{2p} \), the dihedral group with \( 2p \) element. Let \( N \) be the normal cyclic subgroup of order \( p \) in \( G \). Let \( e_N := \frac{1}{p} \sum_{x \in N} x \in \mathbb{Q}[G] \).

(i) Show that \( e_N \in \mathbb{Q}[G] \) and \( e_N^2 = 1 \).

(ii) Let \( \pi : \mathbb{Q}[G] \to \mathbb{Q}[G/N] \cong \mathbb{Q}[\mathbb{Z}/2\mathbb{Z}] \) be the surjective ring homomorphism induced by the canonical surjection \( G \to G/N \). Show that \( (1 - e_N) \) generates \( \text{Ker}(\pi) \).

(iii) The ideal \( (1 - e_N)\mathbb{Q}[G] = (1 - e_N)\mathbb{Q}[G](1 - e_N) = \mathbb{Q}[G](1 - e_N) \) of \( \mathbb{Q}[G] \) has a natural structure as a \( \mathbb{Q} \)-algebra; denote it by \( A \). Show that \( A \) is isomorphic to \( \mathbb{Q}[G]/e_N \mathbb{Q}[G] \) as a \( \mathbb{Q} \)-algebra. Prove that the center of \( \mathbb{Q}[G]/e_N \mathbb{Q}[G] \) is equal to the field \( \mathbb{Q}[N]/e_N \mathbb{Q}[N] := F \).

(iv) Is \( \mathbb{Q}[G]/e_N \mathbb{Q}[G] \) isomorphic to \( M_2(\mathbb{Q}) \)? (Either establish an isomorphism or show that no such isomorphism exists.)

8. Let \( Q = Q_8 \) be the quaternion group with 8 elements. The center \( Z = Z(Q) \) of \( Q \) is generated by the unique element \( \sigma \) of order 2 in \( Q \). Let \( e_Z := \frac{1 + \sigma}{2} \in \mathbb{Q}[Q] \). Denote by \( A \) the \( \mathbb{Q} \)-algebra \( \mathbb{Q}[Q]/e_Z \mathbb{Q}[Q] \).

(i) Show that the center of \( A \) is \( \mathbb{Q} \), i.e. \( \text{dim}_\mathbb{Q}(Z(A)) = 1 \), and \( \text{dim}_\mathbb{Q}(A) = 4 \).

(ii) Is \( A \) isomorphic to \( M_2(\mathbb{Q}) \)? (Either establish an isomorphism or show that no such isomorphism exists.)