Notes Jacobson rings

§1. Definitions and Lemmas

(1.1) Definition An integral domain R is a *Goldman domain* if there exists a finite number of non-zero elements $u_1, \dots, u_n$ such that $R[u_1^{-1}, \dots, u_n^{-1}] = K$, the field of fractions of R. Notice that then $K = R[u^{-1}]$, where $u = \prod_i u_i$.

(1.2) Lemma *Let R be a Goldman domain with fraction field K, S is an R-subalgebra of K. Then S is also a Goldman domain.*

(1.3) Definition Let R be a commutative ring. A prime ideal $P \subset R$ is an *Goldman ideal* if R/P is a Goldman domain.

(1.4) Definition A commutative ring R is said to be a *Jacobson ring* if every Goldman prime ideal is a maximal ideal.

(1.5) Proposition *Let R be a commutative ring and let $u \in R$ be a non-unipotent element of R. Then there exists a Goldman prime ideal P of R which does not contain u.*

Proof. Let $S = u^{\mathbb{N}}$, let $\mathfrak{m}$ be a maximal ideal of $S^{-1}R$, and let $P = \mathfrak{m} \cap R$. Then $R_1 := R/P$ is an integral domain and $R_1[\bar{u}^{-1}] \cong S^{-1}R/\mathfrak{m}]$ is isomorphic to the fraction field of R_1.

(1.6) Corollary *Let I be an ideal in a commutative ring R. Then $\sqrt{I}$ is equal to the intersection of all Goldman prime ideals which contain I.*

(1.7) Lemma *Let R_1 be an integral domain contained in a field L. If L is integral over R_1, then R_1 is a field.*

(1.8) Proposition *Let $R \subset S$ be integral domains such that S is a finitely generated R-algebra which is integral over R. Let K, L be the field of fractions of R, S respectively. Then R is a Goldman domain if and only if S is a Goldman domain.*

Proof. Let $S = R[v_1, \dots v_m]$with $v_i \in S$. Suppose first that $K = R[u^{-1}]$ with $u \in R$. Then $S[u^{-1}]$ is an integral domain which is algebraic over K and generated by $v_1, \dots, v_m$, hence $S[u^{-1}]$ is a field, necessarily equal to L.

Conversely, assume that S is a Goldman domain and $L = S[v^{-1}]$, $v \in S$. Then after adjoining a finite number of elements Let a_i be the leading coefficient of an algebraic equations of v_i over R, $i = 1, \dots, m$; let a be the leading coefficient of an algebraic equation of v^{-1} over R. Let $R_1 = R[a_1^{-1}, \dots, a_m^{-1}, a] \subset L$, an integral domain finitely generated over R. Then $L = R_1[v_1, \dots, v_m, v^{-1}]$, and the R_1-generators $v_1, \dots, v_m, v^{-1}]$ are integral over R_1. Hence R_1 is a field by Lemma 1.7. So R is a Goldman domain. □

(1.9) Proposition *Let R be a commutative ring and let P be a prime ideal of R. Then P is a Goldman ideal if and only if it is the contraction of a maximal ideal in the polynomial ring $R[x]$ (resp. the polynomial ring $R[x_1, \ldots, x_n]$).*

(1.10) Theorem *A commutative ring R is a Jacobson ring if and only if the polynomial ring $R[x]$ is a Jacobson ring.*

PROOF. The "if" part is obvious. Assume now that R is a Jacobson ring and P is a Goldman prime ideal of $R[x]$. Let $Q = P \cap R$. Consider $R_1 := R/Q \subset R[x]/P = R_1[\bar{x}] = R_2$. Since R_2 is a Goldman domain by assumption, so is R_1 by Prop. 1.8. Therefore Q is a maximal ideal of R and R_1 is a field, because R is a Jacobson ring. The domain $R_1[\bar{x}]$ is a quotient of a polynomial ring over the field R_1, hence $R_1[\bar{x}]$ is a field, i.e. Q is a maximal ideal. □

(1.11) Corollary (Nullstellensatz) *Let K be an algebraically field.*

(i) *Every maximal ideal of $K[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $\underline{a} = (a_1, \ldots, a_n) \in K^n$.*

(ii) *Let I be an ideal in $K[x_1, \ldots, x_n]$. Then the radical $\sqrt{I}$ of I consists of all polynomials $f(\underline{x})$ such that $f(\underline{a}) = 0$ for all common zeroes $\underline{a} = (a_1, \ldots, a_n)$ of I.*