Notes on semisimple algebras

§1. Semisimple rings

(1.1) Definition A ring \( R \) with 1 is semisimple, or left semisimple to be precise, if the free left \( R \)-module underlying \( R \) is a sum of simple \( R \)-module.

(1.2) Definition A ring \( R \) with 1 is simple, or left simple to be precise, if \( R \) is semisimple and any two simple left ideals (i.e. any two simple left submodules of \( R \)) are isomorphic.

(1.3) Proposition A ring \( R \) is semisimple if and only if there exists a ring \( S \) and a semisimple \( S \)-module \( M \) of finite length such that \( R \cong \text{End}_S(M) \)

(1.4) Corollary Every semisimple ring is Artinian.

(1.5) Proposition Let \( R \) be a semisimple ring. Then \( R \) is isomorphic to a finite direct product \( \prod_{i=1}^s R_i \), where each \( R_i \) is a simple ring.

(1.6) Proposition Let \( R \) be a simple ring. Then there exists a division ring \( D \) and a positive integer \( n \) such that \( R \cong M_n(D) \).

(1.7) Definition Let \( R \) be a ring with 1. Define the radical of \( R \) to be the intersection of all maximal left ideals of \( R \). The above definitions uses left \( R \)-modules. When we want to emphasize that, we say that \( n \) is the left radical of \( R \).

(1.8) Proposition The radical of a semisimple ring is zero.

(1.9) Proposition Let \( R \) be a simple ring. Then \( R \) has no non-trivial two-sided ideals, and its radical is zero.

(1.10) Proposition Let \( R \) be an Artinian ring whose radical is zero. Then \( R \) is semisimple. In particular, if \( R \) has no non-trivial two-sided ideal, then \( R \) is simple.

(1.11) Remark In non-commutative ring theory, the standard definition for a ring to be semisimple is that its radical is zero. This definition is different from Definition 1.1. For instance, \( \mathbb{Z} \) is not a semisimple ring in the sense of Def. 1.1, while the radical of \( \mathbb{Z} \) is zero. In fact the converse of Prop. 1.10 holds; see Cor. 1.4 below.

(1.12) Exercise. Let \( R \) be a ring with 1. Let \( n \) be the radical of \( R \)

(i) Show that there exists a maximal left ideal in \( R \). Deduce that the radical of \( R \) is a proper left ideal of \( R \). (Hint: Use Zorn’s Lemma.)

(ii) Show that \( n \cdot M = (0) \) for every simple left \( R \)-module \( M \). (Hint: Show that for every \( 0 \neq x \in M \), the set of all elements \( y \in R \) such that \( y \cdot x = 0 \) is a maximal left ideal of \( R \).)

(iv) Suppose that \( I \) is a left ideal of \( R \) such that \( I \cdot M = (0) \) for every simple left \( R \)-module \( M \). Prove that \( I \subseteq n \).
Therefore consider the element $a$.

Condition that $0 = a \otimes b$ for all $a, b$.

Let $I$ be a left ideal of $R$ such that $I^n = (0)$ for some positive integer $n$. Show that $I \subseteq n$.

Show that the radical of $R/n$ is zero.

**Exercise.** Let $R$ be a ring with 1 and let $n$ be the (left) radical of $R$.

(i) Let $x \in n$. Show that $R \cdot (1 + x) = R$, i.e. there exists an element $z \in R$ such that $z \cdot (1 + x) = 1$.

(ii) Suppose that $J$ is a left ideal of $R$ such that $R \cdot (1 + x) = R$ for every $x \in J$. Show that $J \subseteq n$. (Hint: If not, then there exists a maximal left ideal $m$ of $R$ such that $J + m \ni 1$.)

(iii) Let $x \in n$, and let $z$ be an element of $R$ such that $z \cdot (1 + x) = 1$. Show that $z - 1 \in n$. Conclude that $1 + n \subset R^\times$.

(iv) Show that the $n$ is equal to the right radical of $R$. (Hint: Use the analogue of (i)–(iii) for the right radical.)

**§2. Simple algebras**

**Proposition** Let $K$ be a field. Let $A$ be a central simple algebra over $K$, and let $B$ be simple $K$-algebra. Then $A \otimes_K B$ is a simple $K$-algebra. Moreover $Z(A \otimes_K B) = Z(B)$, i.e. every element of the center of $A \otimes_K B$ has the form $1 \otimes b$ for a unique element $b \in Z(B)$. In particular, $A \otimes_K B$ is a central simple algebra over $K$ if both $A$ and $B$ are.

**Proof.** We assume for simplicity of exposition that $\dim_K(B) < \infty$; the proof works for the infinite dimensional case as well. Let $b_1, \ldots, b_r$ be a $K$-basis of $B$. Define the length of an element $x = \sum_{i=1}^r a_i \otimes b_i \in A \otimes B$, $a_i \in A$ for $i = 1, \ldots, r$, to be $\text{Card}\{ i \mid a_i \neq 0 \}$.

Let $I$ be a non-zero ideal in $A \otimes_K B$. Let $x$ be a non-zero element of $I$ of minimal length. After relabelling the $b_i$’s, we may and do assume that $x$ has the form

$$x = 1 \otimes b_1 + \sum_{i=2}^r a_i.$$ 

Consider the element $[a \otimes 1, x] \in I$ with $a \in A$, whose length is less than the length of $x$. Therefore $[a \otimes 1, x] = 0$ for all $a \in A$, i.e. $[a, a_i] = 0$ for all $a \in A$ and all $i = 2, \ldots, r$. In other words, $a_i \in K$ for all $i = 2, \ldots, r$. Write $a_i = \lambda_i \in K$, and $x = 1 \otimes b \in I$, where $b = b_1 + \lambda_2 b_2 + \cdots + \lambda_r b_r \in B$, $b \neq 0$. So $1 \otimes BbB \subseteq I$. Since $B$ is simple, we have $BbB = B$ and hence $I = A \otimes_K B$. We have shown that $A \otimes_K B$ is simple.

Let $x = \sum_{i=1}^n a_i \otimes b_i$ be any element of $Z(A \otimes_K B)$, with $a_1, \ldots, a_r \in A$. We have

$$0 = [a \otimes 1, x] = \sum_{i=1}^r [a, a_i] \otimes b_i$$

for all $a \in A$. Hence $a_i \in Z(A) = K$ for each $i = 1, \ldots, r$, and $x = 1 \otimes b$ for some $b \in B$. The condition that $0 = [1 \otimes y, x]$ for all $y \in B$ implies that $b \in Z(B)$ and hence $x \in 1 \otimes Z(B)$. \[\square\]

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Corollary Let $A$ be a finite dimensional algebra over a field $K$, and let $n = \dim_K(A)$. If $A$ is a central simple algebra over $K$, then

$$A \otimes_K A^{op} \xrightarrow{\sim} \text{End}_K(A) \cong M_n(K).$$

Conversely, if $A \otimes_K A^{op} \twoheadrightarrow \text{End}_K(A)$, then $A$ is a central simple algebra over $K$. \[\square\]

Proof. Suppose that $A$ is a central simple algebra over $K$. By Prop. 2.1, $A \otimes_K A^{op}$ is a central simple algebra over $K$. Consider the map

$$\alpha : A \otimes_K A^{op} \to \text{End}_K(A)$$

which sends $x \otimes y$ to the element $u \mapsto xuy \in \text{End}_K(A)$. The source of $\alpha$ is simple by Prop. 2.1, so $\alpha$ is injective because it is clearly non-trivial. Hence it is an isomorphism because the source and the target have the same dimension over $K$.

Conversely, suppose that $A \otimes_K A^{op} \twoheadrightarrow \text{End}_K(A)$ and $I$ is a proper ideal of $A$. Then the image of $I \otimes A^{op}$ in $\text{End}_K(A)$ is an ideal of $\text{End}_K(A)$ which does not contain $\text{Id}_A$, so $A$ is a simple $K$-algebra. Let $L := Z(A)$, then the image of the canonical map $A \otimes_K A^{op}$ in $\text{End}_K(A)$ lies in the subalgebra $\text{End}_L(A)$, hence $L = K$. \[\square\]

Lemma Let $D$ be a finite dimensional central division algebra over an algebraically closed field $K$. Then $D = K$. \[\square\]

Corollary The dimension of any central simple algebra over a field is a perfect square.

Lemma Let $A$ be a finite dimensional central simple algebra over a field $K$. Let $F \subset A$ be an overfield of $K$ contained in $A$. Then $[F : K] \mid [A : K]^{1/2}$. In particular if $[F : K]^2 = [A : K]$, then $F$ is a maximal subfield of $A$.

Proof. Write $[A : K] = n^2$, $[F : K] = d$. Multiplication on the left defines an embedding $A \otimes_K F \hookrightarrow \text{End}_F(A)$. By Lemma 3.1, $n^2 = [A \otimes_K : F]$ divides $[\text{End}_F(A) : F] = (n^2/d)^2$, i.e. $d^2 \mid n^2$. So $d$ divides $n$. \[\square\]

Lemma Let $A$ be a finite dimensional central simple algebra over a field $K$. If $F$ is a subfield of $A$ containing $K$, and $[F : K]^2 = [A : K]$, then $F$ is a maximal subfield of $K$ and $A \otimes_K F \cong M_n(F)$, where $n = [A : K]^{1/2}$.

Proof. We have seen in Lemma 2.5 that $F$ is a maximal subfield of $A$. Consider the natural map $\alpha : A \otimes_K F \to \text{End}_K(A)$, which is injective because $A \otimes_K F$ is simple and $\alpha$ is non-trivial. Since the dimension of the source and the target of $\alpha$ are both equal to $n^2$, $\alpha$ is an isomorphism. \[\square\]

Proposition Let $A$ be a central simple algebra over a field $K$. Then there exists a finite separable field extension $F/K$ such that $A \otimes_K F \cong M_n(F)$, where $n = [A : K]^{1/2}$.
Proof. It suffices to show that \( A \otimes_K \mathbb{K}^{sep} \cong M_n(\mathbb{K}^{sep}) \). Changing notation, we may assume that \( K = \mathbb{K}^{sep} \). By Wedderburn’s theorem, we know that \( A \cong M_{m}(D) \), where \( D \) is a central division algebra over \( K = \mathbb{K}^{sep} \). Write \( n = md \) and \( [D : K] = d^2 \), \( d \in \mathbb{N} \). Suppose that \( D \neq K \), i.e. \( d > 1 \). Then \( \text{char}(K) = p > 0 \), and every element of \( D \) is purely inseparable over \( K \). There exists a power \( q \) of \( p \) such that \( x^q \in K \) for every element \( x \in D \). Then for the central simple algebra \( B := D \otimes_K \mathbb{K}^{alg} \cong M_n(\mathbb{K}^{alg}) \), we have \( y^q \in \mathbb{K}^{alg} \) for every element \( y \in B \cong M_d(\mathbb{K}^{alg}) \). The last statement is clearly false, since \( d > 1 \). \( \square \)

(2.8) Theorem (Noether-Skolem) Let \( B \) be a finite dimensional central simple algebra over a field \( K \). Let \( A_1, A_2 \) be simple \( K \)-subalgebras of \( B \). Let \( \phi : A_1 \to A_2 \) be a \( K \)-linear isomorphism of \( K \)-algebras. Then there exists an element \( x \in B^\times \) such that \( \phi(y) = x^{-1}yx \) for all \( y \in A_1 \).

Proof. Consider the simple \( K \)-algebra \( R := B \otimes_K A_1^{opp} \), and two \( R \)-module structures on the \( K \)-vector space \( V \) underlying \( B \): an element \( u \otimes a \) with \( u \in B \) and \( a \in A_1^{opp} \) operates either as \( b \mapsto uba \) for all \( b \in V \), or as \( b \mapsto ub\phi(a) \) for all \( b \in V \). Hence there exists a \( \psi \in \text{GL}_K(V) \) such that

\[
\psi(uba) = u\psi(b)\phi(a)
\]

for all \( u, b \in B \) and all \( a \in A_1 \). One checks easily that \( \psi(1) \in B^\times \): if \( u \in B \) and \( u \cdot \psi(1) = 0 \), then \( \psi(u) = 0 \), hence \( u = 0 \). Then \( \phi(a) = \psi(1)^{-1} \cdot a \cdot \psi(1) \) for every \( a \in A_1 \). \( \square \)

(2.9) Theorem Let \( B \) be a \( K \)-algebra and let \( A \) be a finite dimensional central simple \( K \)-subalgebra of \( B \). Then the natural homomorphism \( \alpha : A \otimes_K Z_B(A) \to B \) is an isomorphism.

Proof. Passing from \( K \) to \( \mathbb{K}^{alg} \), we may and do assume that \( A \cong M_n(K) \), and we fix an isomorphism \( A \cong M_n(K) \).

First we show that \( \alpha \) is surjective. Given an element \( b \in B \), define elements \( b_{ij} \in B \) for \( 1 \leq i, j \leq n \) by

\[
b_{ij} := \sum_{k=1}^{n} e_{ki} b e_{jk},
\]

where \( e_{ki} \in M_n(K) \) is the \( n \times n \) matrix whose \( (k, i) \)-entry is equal to 1 and all other entries equal to 0. One checks that each \( b_{ij} \) commutes with all elements of \( A = M_n(K) \). The following computation

\[
\sum_{i,j=1}^{n} b_{ij} e_{ij} = \sum_{i,j,k} e_{ki} b e_{jk} e_{ij} = \sum_{i,j} e_{ii} b e_{jj} = b
\]

shows that \( \alpha \) is surjective.

Suppose that \( 0 = \sum_{i,j=1}^{n} b_{ij} e_{ij}, b_{ij} \in Z_{B}(A) \) for all \( 1 \leq i, j \leq n \). Then

\[
0 = \sum_{k=1}^{n} e_{kl} \left( \sum_{i,j} b_{ij} e_{ij} \right) e_{mk} = \sum_{k=1}^{n} b_{lm} e_{kk} = b_{lm}
\]

for all \( 0 \leq l, m \leq n \). Hence \( \alpha \) is injective. \( \square \)
(2.10) Theorem Let $B$ be a finite dimensional central simple algebra over a field $K$, and let $A$ be a simple $K$-subalgebra of $B$. Then $Z_B(A)$ is simple, and $Z_B(Z_B(A)) = A$.

Proof. Let $C = \text{End}_K(A) \cong M_n(K)$, where $n = [A : K]$. Inside the central simple $K$-algebra $B \otimes_K C$ we have two simple $K$-subalgebras, $A \otimes_K K$ and $K \otimes_K A$. Here the right factor of $K \otimes_K A$ is the image of $A$ in $C = \text{End}_K(A)$ under left multiplication. Clearly these two simple $K$-subalgebras of $B \otimes_K C$ are isomorphic, since both are isomorphic to $A$ as a $K$-algebra. By Noether-Skolem, these two subalgebras are conjugate in $B \otimes_K C$ by a suitable element of $(B \otimes_K C)^\times$, therefore their centralizers (resp. double centralizers) in $B \otimes_K C$ are conjugate, hence isomorphic.

Let’s compute the centralizers first:

$$Z_{B \otimes_K C}(A \otimes_K K) = Z_B(A) \otimes_K C,$$

while

$$Z_{B \otimes_K C}(K \otimes_K A) = B \otimes_K A^{\text{opp}}.$$

Since $B \otimes_K A^{\text{opp}}$ is central simple over $K$, so is $Z_B(A) \otimes_K C$. Hence $Z_B(A)$ is simple.

We compute the double centralizers:

$$Z_{B \otimes_K C}(Z_{B \otimes_K C}(A \otimes_K K)) = Z_{B \otimes_K C}(Z_B(A) \otimes_K C) = Z_B(Z_B(A) \otimes_K K),$$

while

$$Z_{B \otimes_K C}(Z_{B \otimes_K C}(K \otimes_K A)) = Z_{B \otimes_K C}(B \otimes_K A^{\text{opp}}) = K \otimes_K A$$

So $Z_B(Z_B(A))$ is isomorphic to $A$ as $K$-algebras. Since $A \subseteq Z_B(Z_B(A))$, the inclusion is an equality. \(\square\)

§3. Some invariants

(3.1) Lemma Let $K$ be a field and let $A$ be a finite dimensional simple $K$-algebra. Let $M$ be an $(A, A)$-bimodule. Then $M$ is free as a left $A$-module.

(3.2) Definition Let $K$ be a field, $B$ be a $K$-algebra, and let $A$ be a finite dimensional simple $K$-subalgebra of $B$. Then $B$ is a free left $A$-module by Lemma 3.1. We define the rank of $B$ over $A$, denoted $[B : A]$, to be the rank of $B$ as a free left $A$-module. Clearly $[B : A] = \dim_K(B)/\dim_K(A)$ if $\dim_K(A) < \infty$.

(3.3) Definition Let $K$ be a field. Let $B$ be a finite dimensional simple $K$-algebra, and let $A$ be a simple $K$-subalgebra of $A$. Let $N$ be a left simple $B$-module, and let $M$ be a left simple $A$-module.

(i) Define $i(B, A) := \text{length}_B(B \otimes_A M)$, called the index of $A$ in $B$.

(ii) Define $h(B, A) := \text{length}_A(N)$, called the height of $B$ over $A$.

Recall that $[B : A]$ is the $A$-rank of $B_A$, where $B_A$ is the free left $A$-module underlying $B$.

(3.4) Lemma Notation as in Def. 3.3.

(i) $\text{length}_B(B \otimes_A U) = i(B, A) \text{length}_A(U)$ for every left $A$-module $U$.
(ii) \( \text{length}_A(V) = h(B, A) \cdot \text{length}_B(V) \) for every left \( B \)-module \( V \).

(iii) \( \text{length}_B(B_s) = i(B, A) \cdot \text{length}_A(A_s) \).

(iv) \( \text{length}_A(B \otimes_A U) = [B : A] \cdot \text{length}_A(U) \)

(v) \( [B : A] = h(B, A) \cdot i(B, A) \)

**Proof.** Statement (iii) follows from (iv) and the fact that \( B_s \cong B \otimes_A A_s \). To show (v), we apply (i) a simple \( A \)-module \( M \) and get

\[ [B : A] = \text{length}_A(B \otimes_A M) = h(B, A) \text{length}_B(B \otimes_A M) = h(B, A) i(B, A) . \]

Another proof of (iv) is to use the \( A \)-module \( A_s \) instead of a simple \( A \)-module \( M \):

\[ [B : A] \text{length}_A(A_s) = \text{length}_A(B_s) = \text{length}_B(B_s) h(B, A) = h(B, A) i(B, A) \text{length}_A(A_s) . \]

The last equality follows from (iii). □

(3.5) **Lemma** Let \( A \subset B \subset C \) be inclusion of simple algebras over a field \( K \). Then \( i(C, A) = i(C, B) \cdot i(B, A) \), \( h(C, A) = h(C, B) \cdot h(B, A) \), and \( [C : A] = [C : B] \cdot [B : A] \). □

(3.6) **Lemma** Let \( K \) be an algebraically closed field. Let \( B \) be a finite dimensional simple \( K \)-algebra, and let \( A \) be a semisimple \( K \)-subalgebra of \( B \). Let \( M \) be a simple \( A \)-module, and let \( N \) be a simple \( B \)-module.

(i) \( N \) contains \( M \) as a left \( A \)-module.

(ii) The following equalities hold.

\[
\dim_K(\text{Hom}_B(B \otimes_A M, N)) = \dim_K(\text{Hom}_A(M, N)) = \dim_K(\text{Hom}_A(N, M))
= \dim_K(\text{Hom}_B(N, \text{Hom}_A(B, M)))
\]

(iii) Assume in addition that \( A \) is simple. Then \( i(B, A) = h(B, A) \).

**Proof.** Statements (i), (ii) are easy and left as exercises. The statement (iii) follows from the first equality in (ii). □

(3.7) **Lemma** Let \( A \) be a simple algebra over a field \( K \). Let \( M \) be a non-trivial finitely generated left \( A \)-module, and let \( A' := \text{End}_A(M) \). Then \( \text{length}_A(M) = \text{length}_{A'}(A'_s) \), where \( A'_s \) is the left \( A'_s \)-module underlying \( A' \).

**Proof.** Write \( M \cong U^n \), where \( U \) is a simple \( A \)-module. Then \( A' \cong M_n(D) \), where \( D := \text{End}_A(U) \) is a division algebra. So \( \text{length}_{A'}(A'_s) = n = \text{length}_A(M) \). □
(3.8) Proposition Let $K$ be a field, $B$ be a finite dimensional simple $K$-algebra, and let $A$ be a simple $K$-subalgebra of $B$. Let $N$ be a non-trivial $B$-module. Then

(i) $A' := \text{End}_A(N)$ is a simple $K$-algebra, and $B' := \text{End}_B(N)$ is a simple $K$-subalgebra of $A'$.

(ii) $i(A' , B') = h(B, A)$, and $h(A', B') = i(B, A)$.

Proof. The statement (i) is easy and omitted. To prove (ii), we have

$$\text{length}_A(N) = \text{length}_A(A') = i(A', B') \text{length}_{B'}(B') ,$$

where the first equality follows from Lemma 3.7 and the second equality follows from Lemma 3.4 (iii). We also have

$$\text{length}_A(N) = h(B, A) \text{length}_A(N) = h(B, A) \text{length}_{B'}(B')$$

where the last equality follows from Lemma 3.7. So we get $i(A', B') = h(B, A)$. Replacing $(B, A)$ by $(A', B')$, we get $i(B, A) = h(A', B')$. □

§4. Centralizers

(4.1) Theorem Let $K$ be a field. Let $B$ be a finite dimensional central simple algebra over $K$. Let $A$ be a simple $K$-subalgebra of $B$, and let $A' := Z_B(A)$. Let $L = Z(A) = Z(A')$. Then the following holds.

(i) $A'$ is a simple $K$-algebra.

(ii) $A := Z_B(Z_B(A))$.


(iv) $A$ and $A'$ are linearly disjoint over $L$.

(v) If $A$ is a central simple algebras over $K$, then $A \otimes A' \sim B$.

Proof. Let $N$ be a simple $B$-module. Let $D := \text{End}_B(V)$. We have $D \subseteq \text{End}_K(N) \supseteq B$, and $Z(D) = Z(B) = K$. So $D \otimes_K A$ is a simple $K$-algebra, and we have $D \otimes_K A \sim D \cdot A \subseteq \text{End}_K(N) = C$, where $D \cdot A$ is the subalgebra of $\text{End}_K(N)$ generated by $D$ and $A$. So

$$Z_C(D \cdot A) = Z_C(D) \cap Z_C(A) = B \cap Z_C(A) = A'.$$

Hence $A' = \text{End}_{D,A}(N)$ is simple, because $D \cdot A$ is simple. We have proved (i).

Apply Prop. 3.8 (ii) to the pair $(D \cdot A, D)$ and the $D \cdot A$-module $N$. We get

$$[A : K] = [D \cdot A : D] = [B : A']$$

since $Z_C(D) = B$. On the other hand, we have

$$[B : A] \cdot [A : K] = [B : K] = [B : A'] \cdot [A' : K] = [A : K] \cdot [A' : K]$$

so $[B : A] = [A' : K]$. We have proved (iii).
Apply (i) and (iii) to the simple $K$-subalgebra $A' \subseteq B$, we see that $A \subset Z_B(A')$ and $[A : K] = [A' : K]$, so $A = Z_B(Z_B(A))$. We have proved (ii).

Let $L := A \cap A' = Z(A) \subseteq Z(A') = Z(A)$. The last equality follows from (i). The tensor product $A \otimes_L A'$ is a central simple algebra over $L$ since $A$ and $A'$ are central simple over $L$. So the canonical homomorphism $A \otimes_L A' \to B$ is an injection. We have prove (iv). The above inclusion is an equality if and only if $L = K$, because $\dim_L(B) = [L : K] \cdot [A : L] \cdot [A' : L]$.  

**Remark** (1) Statements (i) and (ii) of Thm. 4.1 is the content of Thm. 2.10. The proof in 2.10 uses Noether-Skolem and the fact that the double centralizer of any $K$-algebra $A$ in $\text{End}_K(A)$ is equal to itself. The proof in 4.1 relies on Prop. 3.8.

(2) Statement (v) of Thm. 4.1 is a special case of Thm. 2.9.

**Corollary** (4.2) Let $A$ be a finite dimensional central simple algebra over a field $K$, and let $F$ be a subfield of $A$ which contains $K$. Then $F$ is a maximal subfield of $A$ if and only if $[F : K]^2 = [A : K]$.

**Proof.** Immediate from Thm. 4.1 (iii).

(4.3) **Proposition** Let $A$ be a finite dimensional central simple algebra over $K$. Let $F$ be an extension field of $K$ such that $[F : K] = n := [A : K]^{1/2}$. Then there exists a $K$-linear ring homomorphism $F \hookrightarrow A$ if and only if $A \otimes_K F \cong M_n(F)$.

**Proof.** The “only if” part is contained in Lemma 2.6. It remains to show the “if” part. Suppose that $A \otimes_K F \cong M_n(F)$. Choose a $K$-linear embedding $\alpha : F \hookrightarrow M_n(K)$. The central simple algebra $B := A \otimes_K M_n(K)$ over $K$ contains $C_1 := A \otimes_K \alpha(F)$ as a subalgebra, whose centralizer in $B$ is $K \otimes_K \alpha(F)$. Since $C_1 \cong M_n(F)$ by assumption, $C_1$ contains a subalgebra $C_2$ which is isomorphic to $M_n(K)$. By Noether-Skolem, $Z_B(C_2)$ is isomorphic to $A$ over $K$. So we get $F \cong Z_B(C_1) \subset Z_B(C_2) \cong A$. 

(4.4) **Theorem** Let $K$ be a field and let $B$ be a finite dimensional central simple algebra over $K$. Let $N$ be a non-trivial $B$-module of finite length. Let $A$ be a simple $K$-subalgebra of $B$. Let $A' := Z_B(A)$ be the centralizer of $A$ in $B$. Then we have a natural isomorphism

$$\text{End}_B(N) \otimes_K A' \cong \text{End}_A(N).$$

**Proof.** We know that $A'$ is a simple $K$-algebra, and $R := \text{End}_B(N)$ is a central simple $K$-algebra. So $R \otimes_K A'$ is a simple $K$-algebra. Let $C$ be the image of $R \otimes_K A'$ in $\text{End}_A(N)$; clearly we have $R \otimes_K A' \cong C$. Let $S := \text{End}_K(N)$. Let $C' := \text{End}_C(N)$. We have

$$C' = \text{End}_R(N) \cap \text{End}_A(N) = B \cap Z_S(A') = Z_B(A') = A;$$

the second and the fourth equality follows from the double centralizer theorem. Hence $C = \text{End}_A(N)$, again by the double centralizer theorem. 

(4.5) **Corollary** Notation as in Prop. 4.4. Let $L := Z(A) = Z(A')$. Then $[A \otimes_L Z_B(A)]$ and $[B \otimes_K L]$ are equal as elements of $\text{Br}(L)$.

**Proof.** Take $N = B$, the left regular representation of $B$, in Thm. 4.4.