1. Let $K$ be a finite separable extension field of $k$.
   (a) Show that the character group of $\text{Res}_{K/k}(\mathbb{G}_m)$ is naturally isomorphic to the free $\mathbb{Z}$-module generated by the set of embeddings of $K$ into the separable closure $k^{\text{sep}}$ of $k$.
   (b) Describe the action of $\text{Gal}(k^{\text{sep}}/K)$ on the character group of $\text{Res}_{K/k}(\mathbb{G}_m)$.

2. Let $T$ be a torus over $\mathbb{R}$. Prove that $T(\mathbb{R})$ is compact if and only if the non-trivial element $\tau \in \text{Gal}(\mathbb{C}/\mathbb{R})$ operates on $X^*(T)$ as $-1$.

3. Prove that $T(\mathbb{R})$ is Zariski dense in $T$ for any torus $T$ over $\mathbb{R}$.

4. Let $T$ be a torus over $\mathbb{R}$.
   (i) Prove that for any homomorphism $\nu : \mathbb{G}_m \rightarrow T$ over $\mathbb{C}$, there exists a homomorphism $h : S \rightarrow T$ over $\mathbb{R}$ such that $h \circ \mu = \nu$.
   (ii) If $h_1, h_2 : S \rightarrow T$ are two $\mathbb{R}$-homomorphisms of tori over $\mathbb{R}$ such that $h_1 \circ \mu = h_2 \circ \mu$, then $h_1 = h_2$.

5. Let $T$ be a torus over $\mathbb{R}$. Prove that for any homomorphism $\chi : T \rightarrow \mathbb{G}_m$, there exists exactly one $\mathbb{R}$ homomorphism $h : T \rightarrow S$ such that $\chi_z \circ h = \chi$.

6. Prove that the homomorphism $\text{Nm} \circ \beta : \mathbb{G}_m \times \mathbb{C}^\times \rightarrow \mathbb{G}_m$ is equal to the square morphism on the first factor of the source and is trivial on the second factor.

7. (i) Find all $\mathbb{R}$-endomorphisms of the torus $S$.
   (ii) Find all $\mathbb{R}$-homomorphisms from $S$ to $\mathbb{C}^\times$.

8. Let $V$ be a finite dimensional vector space over $\mathbb{Q}$. Let $h : S \rightarrow \text{GL}(V_\mathbb{R})$ be an $\mathbb{R}$-homomorphism, giving $V_\mathbb{R}$ the structure of a real Hodge structure. Then $(V, h)$ is a $\mathbb{Q}$-Hodge structure if and only if the weight cocharacter

   \[ w_h := h \circ w : \mathbb{G}_m \rightarrow \text{GL}(V_\mathbb{R}) \]

   of $\text{GL}(V_\mathbb{R})$ is defined over $\mathbb{Q}$.

9. Let $V$ be an irreducible $\mathbb{R}$-Hodge structure, and $\dim_{\mathbb{Q}}(V) = 2$ of weight $n$. Prove that the homomorphism $h_V : S \rightarrow \text{GL}(V)$ has the form $h_V = h_1 \circ [n]_S$, where $h_1 : S \rightarrow \text{GL}(V)$ is an $\mathbb{R}$-homomorphism giving $V$ the structure of a real Hodge structure of weight 1.

10. Prove that every one-dimensional $\mathbb{Q}$-Hodge structure is isomorphic to $\mathbb{Q}(i)$ for some $i \in \mathbb{Z}$.  

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11. Give an example of a subgroup \( S \) of \( \text{GL}_n(\mathbb{C}) \) such that the smallest \( \mathbb{Q} \)-subvariety \( Z \) of the \( \mathbb{Q} \)-algebraic group \( \text{GL}_n \) which contains \( S \) is not a stable under multiplication. In other words, \( Z \) is not a \( \mathbb{Q} \)-algebraic subgroup.

(Hint: Here is one example. Take an integer \( m \geq 2 \). Consider the following subsubgroup
\[
S = \left\{ \text{Ad} \left( \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \text{GL}_m(\mathbb{C}) \right\}
\]

of \( \text{GL}_{2m} \), where \( b = (b_{ij}) \) is an element of \( M_m(\mathbb{C}) \) such that \( \text{tr. deg } \mathbb{Q}(n_{ij}) = m^2 \). Then it is not difficult to show that the \( \mathbb{Q} \)-Zariski closure of \( S \) is equal to
\[
\left\{ \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} : \text{tr}(A^i B) = 0, i = 0, 1, \ldots, m-1 \right\}
\]

On the other hand, one can show that the smallest \( \mathbb{Q} \)-subgroup containing \( S \) is equal to
\[
\left\{ \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} \right\}
\]

Notice that this group is not reductive.)

12. Let \( F \) be a totally real number field. Let \( V \) be a two-dimensional vector space over \( F \), with a \( \mathbb{Q} \)-Hodge structure of Hodge type \( \{(0, -1), (-1, 0)\} \) on \( V \), such that the subset \( F \subset \text{End}_{\mathbb{Q}}(V) \) consists of Hodge cycles. In other words, the action of \( F \) on \( V \) preserves the Hodge filtration. Prove that the Mumford-Tate group \( \text{MT}(V) \) is either \( \text{GL}_F(V) \), or is isomorphic to the inverse image of \( \mathbb{G}_m \) under
\[
N_{K/F} : \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \to \text{Res}_{F/\mathbb{Q}}(\mathbb{G}_m),
\]

where \( K \) is a totally imaginary quadratic extension field of \( F \).

13. Let \( V \) be a finite dimensional vector space over \( \mathbb{C} \), which is a direct sum of two nontrivial subspaces \( V_0 \) and \( V_1 \) such that the dimensions \( n_0, n_1 \) of \( V_0, V_1 \) are relatively prime. Let \( G \) be a connected reductive algebraic subgroup of \( \text{GL}(V) \) such that the commutant of \( G \) in \( \text{GL}(V) \) is equal to the group of homotheties on \( V \). Assume moreover that \( G \) contains the subgroup \( H \) of \( \text{GL}(V) \) consisting of all elements which act as homotheties on \( V_0 \) and on \( V_1 \).

(a) Let \( Z \) be the center of \( G \) and let \( G^{\text{der}} \) be the derived group of \( G \). Show that \( Z \) is equal to the group of homotheties on \( V \), and the action of \( G^{\text{der}} \) is irreducible.

(b) For any finite dimensional linear representation \( W \) of the torus \( H \), denote by \( \chi_H(W) \) the character of \( V \); \( \chi_H(W) \) is a linear combination of elements of the character group \( X^{\ast}(H) \) with coefficients in \( \mathbb{Z}_{\geq 0} \). Show that if \( W_1, W_2 \) are \( H \)-modules such that \( \chi_H(V) = \chi_H(W_1) \cdot \chi_H(W_2) \), then either \( W_1 \) or \( W_2 \) is one-dimensional.
(c) Deduce from (b) that $G^{\text{der}}$ is an almost simple semisimple algebraic subgroup of $\text{SL}(V)$.

(d) Use the fact that the center of $G^{\text{der}}$ contains a cyclic subgroup of order $n = \dim(V)$ and the classification of simply connected semisimple algebraic groups to prove that $G^{\text{der}}$ is equal to $\text{SL}(V)$.

14. Let $V$ be a $\mathbb{Q}$-Hodge structure of type $\{(0, -1), (-1, 0)\}$, such that $\text{End}_{\mathbb{Q}\text{-Hdg}}(V)$ contains a imaginary quadratic field $K$. Decompose $V^{-1,0}$ as

$$V^{-1,0} = V_0^{-1,0} \oplus V_1^{-1,0}$$

such that $K$ operates on $V_0^{-1,0}$ via an embedding $\sigma : K \hookrightarrow \mathbb{C}$, and $K$ operates on $V_1^{-1,0}$ via the complex conjugate of $\sigma$. Prove that if the dimensions $n_0, n_1$ of $V_0^{-1,0}, V_1^{-1,0}$ are relatively prime, then the Mumford-Tate group $\text{MT}(V)$ is isomorphic to $\text{GL}(V)$.

(Note: Problems 13 and 14 are taken from Serre, Proc. Cof. Local Fields, Driebergen, 118–131, 1966.)