1. Let $K$ be an imaginary quadratic field. For every $z \in K$ with $\text{Im}(z) > 0$, let $L_z$ be the lattice $\mathbb{Z} + \mathbb{Z} \cdot z \subset \mathbb{C}$, and let $E_z$ be the elliptic curve $\mathbb{C}/L_z$. Denote by $j(z) = j(E_z)$ the $j$-invariant of $E_z$.

(i) The ring of endomorphisms $\text{End}(E_z)$ of $E_z$ is an order of $\mathcal{O}_K$, necessarily equal to $\mathbb{Z} + f\mathcal{O}_K$ for a unique $f \in \mathbb{N}_{>0}$. Show that $K(j(z))$ is the ring class field of $K$ with conductor $f$, i.e. the abelian extension of $K$ which corresponds to the subgroup

$$(\mathbb{Z} + f\mathcal{O}_K) \otimes \mathbb{Z} \cdot \widehat{\mathbb{Z}} / (K_\infty^\times \cdot K^\times) \subset \mathcal{A}_K^\times / (K_\infty^\times \cdot K^\times)$$

under class field theory.

(ii) Let $K^\dagger$ be the extension field of $K$ generated by $\mathbb{Q}^{cyc}$ and all elements of the form $j(z)$, with $z \in K$, $\text{Im}(z) > 0$. Describe the abelian extension $K^\dagger$ of $K$ using class field theory, and show that $\text{Gal}(K_{ab}^\dagger/K^\dagger)$ is a product of groups of order 2.

2. Let $k$ be a number field. Denote by $k^S$ the quotient of $k^\times$ by $\text{Res}_{k/\mathbb{Q}}\mathbb{G}_m$ such that

$$X^*(k^S) = \{ \chi \in X^*(k^\times) \mid (1 - \sigma)(1 + \iota)\chi = (1 + \iota)(1 - \sigma)\chi = 0, \ \forall \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \}$$

where $\iota$ denotes a complex conjugation. Prove that there exists a subgroup $\Gamma \subset \mathcal{O}_K^\times$ of finite index such that for every subgroup $\Gamma_1 \subset \Gamma$ of finite index, the quotient of $k^\times$ by the Zariski closure of $\Gamma_1$ is equal to $k^S$.

3. Let $k$ be a number field. Recall that a $\mathbb{C}^\times$-valued algebraic Hecke character

$$\psi : \mathbb{A}_k^\times \rightarrow \mathbb{C}^\times$$

is a continuous character such that the restriction

$$\psi_{\infty,+} : k_{\infty,+}^\times = (k \otimes \mathbb{R})_+^\times \rightarrow \mathbb{C}^\times$$

of $\psi$ to the neutral component $k_{\infty,+}$ of $k_\infty = k \otimes \mathbb{R}$ coincides with the restriction to $k_{\infty,+}$ of a character $\chi_\psi$ of $k^\times$ defined over $\mathbb{C}$; sometimes $\chi_\psi$ is called the infinity type of $\psi$. Use the problem 2 above to show the following.

(i) For every algebraic Hecke character $\psi$ of $\mathbb{A}_k^\times$, the infinity component $\psi_{\infty}$ factors through the quotient $k^\times \rightarrow k^S$.

(ii) If two algebraic Hecke characters $\psi, \psi'$ have the same infinity component, then $\psi' \cdot \psi^{-1}$ has finite order.

(iii) Every character of $k^S$ is the infinity component of an algebraic Hecke character of $\mathbb{A}_k^\times$. 
4. We use the geometric normalization for the reciprocity law
\[ \text{rec}_Q : \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times \to \text{Gal}(\mathbb{Q}_{ab}^\times / \mathbb{Q}). \]

Let \( \chi_{\text{cyc}} : \text{Gal}(\mathbb{Q}_{ab}^\times / \mathbb{Q}) \to (\hat{\mathbb{Z}})^\times \) be the cyclotomic character coming describing the action of \( \text{Gal}(\mathbb{Q}_{ab}^\times / \mathbb{Q}) \) on the torsion points of \( \mathbb{G}_m \).

(i) Show that \( \chi_{\text{cyc}} \circ \text{rec}_Q \) can be uniquely extended to an algebraic Hecke character
\[ \psi_{\text{cyc}} : \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times \to \hat{\mathbb{Z}}^\times \times \mathbb{R}^\times \subset \mathbb{A}_{\mathbb{Q}}^\times \]
such that \( \psi_{\text{cyc}} \) is the product of a continuous homomorphism \( c : \mathbb{A}_{\mathbb{Q}}^\times \to \hat{\mathbb{Z}}^\times \) and a homomorphism \( \mathbb{G}_m(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{G}_m(\mathbb{A}_{\mathbb{Q}}) \) coming from a character \( \chi \) of \( \mathbb{G}_m \).

(ii) Show that the restriction of \( \psi_{\text{cyc}} \) to \( \mathbb{Q}^\times \) is
\[ \psi_{\text{cyc}}|_{\mathbb{Q}^\times} : a \mapsto (p^{\text{ord}_p(a)}, p^{\text{ord}_p(a)} a, p^{\text{ord}_p(a)}) \in (\prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell^\times) \times \mathbb{Z}_p^\times \times \mathbb{R}^\times \]
and the restriction of \( \psi_{\text{cyc}} \) to \( \mathbb{R}^\times \) is
\[ \psi_{\text{cyc}}|_{\mathbb{R}^\times} : a \mapsto (\text{sgn}(a), |a|) \in \hat{\mathbb{Z}}^\times \times \mathbb{R}^\times \]

5. (This problem is due to Shimura-Taniyama.) Let \( C \) be the projective completion of the affine curve over \( \mathbb{Q} \) given by the equation \( y^2 = x^p - 1 \), where \( p \) is an odd prime number. Let \( \mathbb{Q}(\mu_p) \) be the cyclotomic field generated by the \( p \)-th roots of unity, and let \( \mathbb{Z}[\mu_p] = \mathbb{Q}(\mu_p) \). There is an action of \( \mu_p \) on \( C \) defined over \( \mathbb{Q}_p \), given by
\[ \zeta_p : (x, y) \to (\zeta_p x, y) \]
where \( \zeta_p \) is a primitive \( p \)-th root of unity. This action gives an action of \( \mu_p \) on the Jacobian \( \text{Jac}(C) =: A \) of \( C \), so that \( A \) is an abelian variety with CM by \( \mathbb{Z}[\mu_p] \).

(i) Prove that the genus of \( C \) is \( g = \frac{p-1}{2} \).

(ii) Show that for \( i = 1, \ldots, g \), the meromorphic differential forms \( \omega_i = \frac{x^{i-1} dx}{y} \) extends to regular differential forms on \( C \) and form a basis of \( \text{H}^0(C, K_C) \).

(iii) The Galois group \( \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \) is canonically isomorphic to \( (\mathbb{Z}/p\mathbb{Z})^\times \), and consists of elements
\[ \sigma_i : \zeta_p \mapsto \zeta_p^i, \quad i \in (\mathbb{Z}/p\mathbb{Z})^\times \]
Show that the CM-type of \( A \) is equal to \( \{ \sigma_1, \ldots, \sigma_g \} \) and that the reflex field is equal to \( \mathbb{Q}(\mu_p) \).

(iv) Deduce from the theory of complex multiplication that for every ideal \( a \subset \mathbb{Z}[\mu_p] \), there exists an element \( x \in \mathbb{Q}(\mu_p) \) such that \( N_{\mu}(a) = (x) \) and \( N(a) = x\bar{x} \), where \( N_{\mu} \) denotes the reflex type norm. Note that this is a special case of the Stickelberger relation, see Lang’s *Algebraic Number Theory*, Chap. 4, §4, Theorem 11.