EXERCISE 3A, 10/9/2005

CORRECTION TO EXERCISE 3 AND DISCUSSION OF RELATED ISSUES

   (i) Statements (i), (ii) of are wrong. In fact the abscissa of convergence for \( F(s) \) is at most \( \frac{1}{2} \); see Problem 3 below.
   (ii) There is a closed formula for the coefficients \( b_n \) of \( F(s) = \left( \sum_{n \geq 1} (-1)^{n-1} n^{-s} \right)^2 = \sum_{n \geq 1} b_n n^{-s} \):
   \[
   b_n = \begin{cases} 
   d(n) & \text{if } n \text{ is odd} \\
   -3d(n) + 4d(n/2) = (a - 3)d(m) & \text{if } n = 2^a m, \ a \geq 1, \ m \text{ odd}
   \end{cases}
   \]
   (iii) \( \sum_{n \text{ odd}, n \leq N} d(n) = \frac{1}{4} N \log N + O(N) \).
   (iv) It would be interesting to determine the abscissa of convergence of \( F(s) \).

2. Show that if \( \sum_{n \geq 1} a_n n^{-s} \) is absolutely convergent at \( s = s_0 \) and \( \sum_{n \geq 1} b_n n^{-s} \), then their formal product is also convergent at \( s = s_0 \).

3. Consider two Dirichlet series \( f(s) = \sum_{n \geq 1} a_n n^{-s} \) and \( g(s) = \sum_{n \geq 1} b_n n^{-s} \). Let
   \[
   h(s) = \sum_{n \geq 1} u_n n^{-s}, \quad u_n = \sum_{d \mid n} a_d u_{n/d} \quad \forall n \geq 1
   \]
   be the formal product of \( f(s) \) and \( g(s) \). Let
   \[
   A(n) = \sum_{m \leq n} a_n, \quad B(n) = \sum_{m \leq n} b_n, \quad U(N) = \sum_{n \leq N} u_n.
   \]
   (i) Suppose that \( f(s) \) converges at \( s = s_1 \) with \( \text{Re}(s_1) = \epsilon_1 > 0 \) and \( g(s) \) converges at \( s = s_2 \) with \( \text{Re}(s_2) = \epsilon_2 > 0 \). Show that there exist positive numbers \( C_1, D_1, C_2, D_2 \) such that
   \[
   |A_n| \leq C_1 n^{\epsilon_1}, \ |a_n| \leq D_1 n^{\epsilon_1}, \ |B_n| \leq C_2 n^{\epsilon_2}, \ |b_n| \leq D_2 n^{\epsilon_2}.
   \]
   (ii) Notation as in (i) above. Show that
   \[
   |U(N)| \leq 2(C_1 D_2 + C_2 D_1) N^{1 + \epsilon_1 + \epsilon_2}
   \]
   (iii) Assume that \( f(s) \) and \( g(s) \) are both convergent for \( \text{Re}(s) > 0 \). Prove that \( \sum_{n \geq 1} u_n n^{-s} \) is convergent for \( \text{Re}(s) > \frac{1}{2} \).

4. Let \( f(s) = \sum_{n \geq 1} (-1)^{n-1} (\log 2n)^{-2} n^{-s} \), and let \( F(s) = f(s)^2 = \sum_{n \geq 1} b_n n^{-s} \).
   (i) Show that \( f(s) \) is convergent at \( s = 0 \).
   (ii) Find the abscissa of convergence and the abscissa of absolute convergence for \( f(s) \)
   (iii) Show that \( F(s) \) diverges at \( s = 0 \).
   (Hint: Show that the \( b_n \)'s are unbounded.)