Exercise Set 1

A = a Dedekind domain, not a field

$K^0(A) :=$ the Grothendieck K-group attached to the category of all finitely generated projective A-modules

$K_0(A) :=$ the Grothendieck K-group attached to the category of all finitely generated A-modules

$rk : K^0(A) \twoheadrightarrow \mathbb{Z}$, $rk : K_0(A) \twoheadrightarrow \mathbb{Z}$ are the homomorphisms given by the rank of finitely generated A-modules

1. Show that the kernel of every surjection $A^{\oplus m} \xrightarrow{\beta} M$ of A-modules is a finitely generated projective A-module.

2. Show that the natural map $K^0(A) \longrightarrow K_0(A)$ is an isomorphism of abelian groups

3. Recall that $K(\mathcal{C}(A))$ is the K-group attached to the category of all finite length A-modules. Show that the natural map $K(\mathcal{C}(A)) \longrightarrow K_0(A)^{\circ}$ is surjective

4. Define $K^0(A)^{\circ} := \mathrm{Ker}(rk : K^0(A) \to \mathbb{Z})$. Show that the map

$$\begin{array}{ccc} I_A & \longrightarrow & K^0(A)^{\circ} \\ \Downarrow & & \Downarrow \\ J & \longmapsto & \langle J \rangle - \langle A \rangle \end{array}$$

$\langle J \rangle :=$ the element in $K^0(A)$ represented by the invertible A-submodule $J \subseteq \mathrm{frac}(A)$

induces a group isomorphism

$$H_A \xrightarrow[\delta]{\sim} K^0(A)^{\circ}$$

8.(a) Show that for any two finite projective resolutions

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

$$0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \to 0$$

by finitely generated projective A-modules, there exists a <u>functorial</u> isomorphism

$$\beta : \bigotimes_{i=0}^{m} \Lambda^{top}(P_i)^{(-1)^{i-1}} \xrightarrow{\sim} \bigotimes_{j=0}^{n} \Lambda^{top}(Q_j)^{(-1)^{j-1}}$$

Therefore we get a functor

$$\det : \{\text{finitely generated } A\text{-modules}\} \longrightarrow \left\{\begin{matrix}\text{rank-one projective} \\ A\text{-modules}\end{matrix}\right\}$$

(b) Show that for every finite length A-module M, $\det(M)$ is isomorphic to $\tilde{c}_1([M])$

10.* Find an example of a Dedekind domain A and a finite extension field L of $\mathrm{frac}(A)$ such that the integral closure B of A in L is NOT a finitely generated A-module

5. Show that there is a natural ring structure ring structure on $K^0(A)$ where tensor product gives the multiplication. In other words, for any two finitely generated projective A-modules P, P', the product $\langle P\rangle \cdot \langle P'\rangle$ in $K^0(A)$ is $\langle P \otimes_A P'\rangle$

6. (a) Show that $K^0(A)^0$ is an ideal in $K^0(A)$, and $K^0(A) = K^0(A)^0 \oplus \mathbb{Z}\cdot\langle A\rangle$

(b) Show that the product of any two elements in $K^0(A)^0$ is 0.

(Thus the ring structure of $K^0(A)$ is not very interesting.)

7.(a) Show that there is a commutative diagram

$$\begin{tikzcd}
K(\mathcal{C}(A)) \arrow[r] \arrow[d, "\gamma"'] & K_0(A) \\
K^0(A)^0 \arrow[r, "\subseteq"] & K^0(A) \arrow[u, "\simeq"']
\end{tikzcd}$$

where the top horizontal arrow is the natural map from $K(\mathcal{C}(A))$ to $K_0(A)$

(b) Show that the homomorphism γ in 7(a) is a surjection. and the following diagram

$$\begin{tikzcd}
K(\mathcal{C}(A)) \arrow[r, "\gamma"] \arrow[d, "\wr", "\tilde{c}_1"'] & K^0(A)^0 \\
I_A \arrow[r] & H_A \arrow[u, "\wr"', "\delta"]
\end{tikzcd}$$

δ as in exer. 4

commutes, where the bottom horizontal arrow is the natural homomorphism from I_A to H_A. We recall that $\tilde{c}_1 : K(\mathcal{C}(A)) \longrightarrow I_A$

is defined by $\tilde{c}_1([A/\mathfrak{p}]) = \mathfrak{p}^{-1}$ for every maximal ideal $\mathfrak{p} \subseteq A$