MATH 620 EXERCISE SET 3, FALL 2016

Part I.

1. (a) Determine explicitly the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$.
(b) Compute $\operatorname{disc}\big(\mathbb{Q}(\sqrt[3]{2}/\mathbb{Q})\big)$ and $\mathscr{D}\big(\mathbb{Q}(\sqrt[3]{2}/\mathbb{Q})\big)$.

2. Let $F = \mathbb{Q}[T]/(T^3+T+1)$. Find the ring of integers $\mathscr{O}_F$ of this number field F and compute $\operatorname{disc}(F/\mathbb{Q})$.

3. Let p be an odd prime number. Let $\mathbb{Q}(\mu_p)$ be the cyclotomic field generated by a non-trivial p-th root of unity in $\mathbb{C}$.

(a) Show that $\mathbb{Q}(\mu_p)$ contains a unique quadratic subfield, i.e. a subfield of degree 2 over $\mathbb{Q}$.

(b) Prove that $\mathbb{Q}\left(\sqrt{\left(\frac{-1}{p}\right)\cdot p}\right)$ is the quadratic subfield in $\mathbb{Q}(\mu_p)$. Here $\left(\frac{?}{?}\right)$ is the Legendre symbol.

4. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. Denote by $|\ |_p$ the unique extension to $\overline{\mathbb{Q}}_p$ of the normalized absolute value of $\mathbb{Q}_p$. For every positive real number a, let $S_a = \{x \in \overline{\mathbb{Z}_p} : |x|_p < a$.

(a) Show that S_a is an ideal of $\overline{\mathbb{Z}_p}$ for every positive real number a.

(b) Show that the ideal S_a is *not* a finitely generated ideal of $\overline{\mathbb{Z}_p}$.

(c) Besides the ideals S_a's, are there other non-finitely generated ideals of $\overline{\mathbb{Z}_p}$? Either give such an example, or show that no such example exists.

5. Let N be a positive integer.

(a) Show that up to isomorphism there are only a finite number of extension fields of $\mathbb{Q}_p$ of degree at most N.

(b) Is it true that to isomorphism there are only a finite number of extension fields of $\mathbb{F}_p((t))$ of degree at most N?

Part II. The problems below are geared toward a better understanding of the different and its relation to Kähler differentials for more general extension of rings.

It is well-known that for a finite extension of complete discrete valuation rings $A \to B$ with separable residue field extension, the quotient $B/\mathscr{D}_{B/A}$ is a cyclic B-module isomorphic to the B-module of relative Kähler differentials $\Omega(B/A)$. The proofs you find in books generally go by direct computation: there exists a presentation $A[T]/f(T)A[T] \cong B$ for some $f(T) \in A[T]$. It is immediate that and $\Omega(B/A)$ is isomorphic to $B[T]/(f'(T))$, while we have seen in class that the inverse different $\mathscr{D}_{B/A}^{-1}$ is the principal fractional B-ideal generated by $f'(T)$.

One question I asked myself a few days ago is this: whether one can find an arrow (or a diagram of arrows) which "explains" this above isomorphism. A related question is that, for more general ring extensions, whether there is a reasonable definition of "different" so that we can relate it to the relative Kähler differentials.

There may not be a definitive answer to this question. The problems below provide a partial answer. You may be able to find a better answer, and please share your success with me.

6. Let $R \to S$ be an injective homomorphism of integral domains such that S is a finite R-module.

(a) Show that there exists a unique R-homomorphism

$$\phi : S \otimes_R S \to \mathrm{Hom}_R(\mathrm{Hom}_R(S,R),R)$$

which satisfies

$$\phi(\sum_i a_i \otimes b_i)(h) = \sum_i a_i h(b_i)$$

for all $\sum_i a_i \otimes b_i) \in S \otimes_R S$ and for all $h \in \mathrm{Hom}_R(S,R)$

(b) Show that ϕ is an isomorphism if S is a finite projective R-module.

(c) Let I be the kernel of the multiplication map $\mu_S : S \otimes_R S \to S$ for S. Let $\mathrm{Ann}_{S\otimes_R S}(I)$ be the annhilator ideal of I in $S \otimes_R S$. Show that $\phi(\mathrm{Ann}_{S\otimes_R S}(I)) \subset \mathrm{Hom}_S(\mathrm{Hom}_R(S,R),R)$, where $\mathrm{Hom}_R(S,R)$ is given the natural S-module structure, i.e. $(a \cdot h)(b) = h(ab)$ for all $a, b \in S$ and all $h \in \mathrm{Hom}_R(S,R)$.

(d) If S is a finite projective R-module, then ϕ induces an R-linear isomorphism from $\mathrm{Ann}_{S\otimes_R S}(I)$ to $\mathrm{Hom}_S(\mathrm{Hom}_R(S,R),R)$.

7. Suppose that the fraction field L of S is a finite separable field extension of the fraction field K of R. Let $\mathrm{tr}_{L/K} \in \mathrm{Hom}_K(L,K)$ be the L/K-trace. Assume that $S \cdot \mathrm{tr}_{L/K} \subset \mathrm{Hom}_S(S,R)$.

(a) Let I_K be the kernel of the multiplication map homomorphism $\mu_L : L \otimes_K L \to L$. Let ϕ_K be the isomorphism from $L \otimes_K L$ to $\mathrm{Hom}_L(\mathrm{Hom}_K(L,K),K)$. Let $\alpha : \mathrm{Ann}_{L\otimes_K L}(I_K) \xrightarrow{\sim} L$ be the L-linear isomorphism between one-dimensional L-vector spaces such that $\alpha(h) = h(\mathrm{tr}_{L/K})$ for all $h \in \mathrm{Hom}_K(L,K)$. Show that the restriction to I_K of the composition $\alpha \circ \phi_K$ is equal to the restriction to I_K of the multiplication map $\mu_L : L \otimes_K L \to L$.

(b) Let $\mathscr{J}$ be the S-submodule of the one-dimensional L-vector space $\mathrm{Hom}_K(L,K)$ such that $\mathscr{J} \cdot \mathrm{tr}_{L/K} = \mathrm{Hom}_R(S,R)$. Define

$$\mathscr{D}_{B/A} := s(S : \mathscr{J}) = \{x \in S \mid x \cdot \mathscr{J} \subset S\}.$$

Show that

$$\mu_S(\mathrm{Ann}_{S\otimes_R S}(I)) \subset \mathscr{D}_{B/A}.$$

(c) Suppose that S is a finite projective R-module. Show that

$$\mu_S(\mathrm{Ann}_{S\otimes_R S}(I)) = \mathscr{D}_{B/A}.$$

8. The module $\Omega(B/A)$ of relative Kähler differentials for B/A is by definition the S-module I/I^2. (You can find standard properties about $\Omega(B/A)$ in all standard books on commutative algebra and/or algebraic geometry. Suppose that S is a finite projective R-module and L over K is separable. Construct a natural S-module homomorphism

$$\beta : \mathscr{D}_{B/A} \to \mathrm{Ann}_S(\Omega(B/A))$$

and show that β is an isomorphism if there is an R-linear ring isomorphism $R[T]/(f(T)) \xrightarrow{\sim} S$, where $f(T) \in R[T]$ is a monic polynomial with coefficients in R.

9. (a) Find an example of an extension of integral domains $R \to S$ with S finite free over R such that β is not an isomorphism.

(b) Are there circumstances more general than the case covered in problem 8 above where you can something about the relation between $\mathscr{D}_{B/A}$ and $\Omega(B/A)$?