MATH 620 EXERCISE SET 4, FALL 2016

1. Let K be a number field. The questions below are related to the above property shown in class.

> *The subset $\mathbb{A}_{K,1}^{\times} \subset \mathbb{A}_K$ is closed in $\mathbb{A}_K$. Moreover the topology on $\mathbb{A}_{K,1}^{\times}$ induced by the inclusion $\mathbb{A}_{K,1}^{\times} \hookrightarrow \mathbb{A}_K^{\times}$ coincides with the topology induced by the inclusion $\mathbb{A}_{K,1}^{\times} \hookrightarrow \mathbb{A}_K$.*

The proof is based on the following (easy) fact: for any positive number $c < 1$, there exists a finite subset $S \subset \Sigma_K$ which contains $\Sigma_{K,\infty}$ such that $||x||_v < c$ for all $v \notin S$ and all $x \in \mathfrak{m}_v^{\wedge}$.

(i) The infinite product of normalized absolute values defines a function

$$||\cdot|| : \mathbb{A}_K \to [0,\infty), \quad (x_v)_{v\in\Sigma_k} \mapsto \prod_{v\in\Sigma_K} ||x_v||$$

on $\mathbb{A}_K$, where $||\cdot||_v$ is the normalized absolute value on $K_v^{\wedge}$. Show that it is upper semi-continuous but not lower semi-continuous.

(ii) Show that for every positive number $a > 0$, the set

$$\{x \in \mathbb{A}_K^{\times} : ||x|| \geq a\}$$

is a closed subset of $\mathbb{A}_K$.

(iii) Show that for every positive number $a > 0$, the set

$$\{x \in \mathbb{A}_K^{\times} : ||x|| \leq b\}$$

is *not* a closed subset of $\mathbb{A}_K$.

(iv) Show that for every two positive numbers a, b with $a < b$, the set

$$\{x \in \mathbb{A}_K^{\times} : a \leq ||x|| \leq b\}$$

is a closed subset of $\mathbb{A}_K$.

2. Let K be a number field.

(a) Let $K^{\times} \cdot \prod_{v\in\Sigma_{K,f}} \mathscr{O}_{K_v}^{\times}$ be the subgroup of $\mathbb{A}_{K,f}^{\times} = \prod'_{v\in\Sigma_{K,f}} K_v^{\times}$ generated by $K^{\times}$ and $\prod_{v\in\Sigma_{K,f}} \mathscr{O}_{K_v}^{\times}$. Show that $K^{\times} \cdot \prod_{v\in\Sigma_{K,f}} \mathscr{O}_{K_v}^{\times}$ is a *closed* subgroup of $\mathbb{A}_{K,f}^{\times}$.

(b) Assume that the subring of algebraic integers $\mathscr{O}_K$ in K is a principal ideal domain. Is $\mathbb{A}_{K,f}^{\times}$ equal to $K^{\times} \cdot \prod_{v\in\Sigma_{K,f}} \mathscr{O}_{K_v}^{\times}$? Either give a proof or give a counter-example.

3. Let K be a global field. Let $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n) \in K[x_1,\ldots,x_n]$ be a system of m polynomials in n variables $x_1,\ldots,x_n$. Let $V(K) = \{(a_1,\ldots,a_n) \in K^n \,|\, f_i(a_1,\ldots,a_n) = 0\ \forall i = 1,\ldots,m\}$. Let $V(\mathbb{A}_K) = \{(a_1,\ldots,a_n) \in \mathbb{A}_K^n \,|\, f_i(a_1,\ldots,a_n) = 0\ \forall i = 1,\ldots,m\}$, endowed with the subspace topology induced from the product topology of $\mathbb{A}_K^n$. Is $V(K)$ a *discrete* subset of $V(\mathbb{A}_K)$? Either give a proof or give a counter-example.

4. Let $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$ be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{A}_{\mathbb{Q}}$ and $ad - bc = 1$, endowed with the topology as a subspace of $\mathbb{A}_K^4$.

(a) Show that $\mathrm{SL}_2(\mathbb{Q})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$

(b) Show that $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})/\,\mathrm{SL}_2(\mathbb{Q})$ is *non-compact*.

(c) Show every left Haar measure on $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$ is a right Haar measure.

(d) Show that $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})/\,\mathrm{SL}_2(\mathbb{Q})$ has *finite* measure for any Haar measure on $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$.