MATH 620 EXERCISE SET 5, FALL 2016

The purpose of this set of problems on Haar measures on a locally compact topological group G is twofold.

(a) Relate left-invariant Haar measures on G to right invariant Haar measures on G.

(b) Discuss existence of left invariant Haar measures on homogenous spaces G/H for closed subgroups $H \subset G$.

We will not need (a) this semester, for the locally compact topological groups we want to do Harmonic analysis will be abelian, but we will need (b).

1. Let G be a locally compact topological group and let $\mu = \mu_G$ be a *left*-invariant Haar measure on G.

(a) Show that there is a unique group homomorphism

$$\Delta = \Delta_G : G \to \mathbb{R}_{>0}^{\times}$$

such that

$$\int_G f(xt^{-1})\, d\mu(x) = \Delta(t) \cdot \int_G f(x)\, d\mu(x)$$

for all $t \in G$ and all $f \in C_c(G)$. (Recall that $C_c(G)$ is the set of all continuous functions with compact support on G. Equivalently $\Delta(t)$ is defined by $\mu(U\,t) = \Delta(t)\,\mu(U)$ for all compact subset $U \subset G$.)

(b) Prove that Δ is continuous.

2. Notation as in problem 1 above.

(a) Show that $\Delta(x)^{-1}\mu_G$ is a right-invariant Haar measure on G. Equivalently

$$f \;\mapsto\; \int_G f(x)\,\Delta(x)^{-1}\, d\mu_G(x) \quad f \in C_c(G)$$

is a right-invariant Haar integral on G.

(b) Show that

$$f \;\mapsto\; \int_G f(x^{-1})\, d\mu_G(x) \quad f \in C_c(G)$$

is a right-invariant Haar integral on G.

(c) Show that

$$\int_G f(x)\,\Delta(x)^{-1}\, d\mu_G(x) = \int_G f(x^{-1})\, d\mu_G(x)$$

for all $f \in C_c(G)$.

[Hint: Let c be the positive real number such that $\int_G f(x)\,\Delta(x)^{-1}\, d\mu_G(x) = c \cdot \int_G f(x^{-1})\, d\mu_G(x)$ for all $f \in C_c(G)$. For every $\varepsilon > 0$, choose an open neighborhood of $1 \in G$ such that $|\Delta(x) - 1|\varepsilon$ for all $x \in U$, and choose a function $f(x)$ with compact support contained in U and $f(x) \geq 1$ on an open neighborhood of 1. Integrate and conclude that $|c - 1| \leq \varepsilon$.]

3. Let H be a closed subgroup of a locally compact group G. The quotient G/H with the quotient topology is a locally compact space. Moreover we have a continuous transitive left action $G \times G/H \to G/H$ of G on G/H. Let μ_H be a left-invariant Haar measure on H.

(a) Let $f \in C_c(G)$. Show that

$$\bar{x} = xH \ \mapsto \int_H f(xy)\, d\mu_H(y)$$

is a well-defined continuous function $\bar{f}$ on G/H with compact support.

(b) Show that for every function $\psi \in C_c(G/H)$, there exists a function $f \in C_c(G/H)$ such that

$$\psi(\bar{x}) = \int_H f(xy)\, d\mu_H(y) \quad \forall x \in G.$$

[Hint: Choose a compact subset $T \subset G$ whose image in G/H contains an open neighborhood of the support of h. Let F be a non-negative function with compact support in G such that $F(t) = 1$ for all $t \in T$. Show that there exists a continuous function ϕ on G/H with such that $\phi(\bar{x}) \cdot \bar{F}(\bar{x}) = \psi(\bar{x})$ for all $\bar{x} \in G/H$. Conclude that the function $x \mapsto \phi(\bar{x}) \cdot F(x)$ has the required property.]

(c) Suppose that we have a left-G-invariant Borel measure $\bar{\nu} = \bar{\nu}_{G/H}$ on G/H, i.e.

$$\int_{G/H} \psi(\bar{x})\, d\bar{\nu}(\bar{x}) = \int_{G/H} \psi(s^{-1}\bar{x})\, d\bar{\nu}(\bar{x}) \quad \forall s \in G,\ \forall \psi \in C_c(G/H).$$

Show that

$$f \ \mapsto \int_{G/H} \bar{f}(\bar{x})\, d\nu_{G/H}(\bar{x}) = \int_{G/H} d\nu_{G/H} \int_H f(xy) d\mu_H(y) \quad f \in C_c(G)$$

is a left-invariant Haar measure on G and

$$\Delta_G(y) = \Delta_H(y) \quad \forall y \in H.$$

4. Let H be a closed subgroup of a locally compact topological group G. Let μ_G, μ_H be left-invariant Haar measures on G and H respectively, and let Δ_G, Δ_H be the moduli for G and H. Assume that $\Delta_G(y) = \Delta_H(y)$ for all $y \in H$.

(a) Suppose that f, F are continuous function with compact support on G. Use Fubini theorem and problem 2 to show that

$$\int_G F(x)\, d\mu_G(x) \int_H f(xy)\, d\mu_H(y) = \int_G f(x)\, d\mu_G(x \int_H F(xy)\, d\mu_H(y).$$

(b) Use 4(a) and 3(b) to show that

$$\int_G f(x)\, d\mu_G(x) = 0$$

if $f \in C_c(G)$ and $\int_G f(xy)\, d\mu_H(y) = 0$ for all $x \in G$.

(c) Prove that there exists a continuous left G-invariant Borel measure on G/H.

[Hint: Use 3(b) and 4(b) to produce a well-defined linear functional $\bar{f} \mapsto \int_G f(x)\, d\mu_G(x)$ on $C_c(G/H)$; show that it defines a Haar measure.]