# MATH 620 EXERCISE SET 6, FALL 2016

1. Let $F$ be a number field. Let $L \supset F$ be a normal closure of $F$ over $\mathbb{Q}$. Let $n = [F : \mathbb{Q}]$. Let $\mathscr{O}_F$, $\mathscr{O}_L$ be the ring of algebraic integers of $F$ and $L$ respectively. Let $\{\alpha_i : F \to L \mid i = 1, \ldots, n\}$ be the set of all ring homomorphisms from $F$ to $L$, and we assume that $\alpha_1$ is the inclusion map.

(a) Let $I$ be the ideal of $\mathscr{O}_L$ generated by all elements of the form $\alpha_1(x) - \alpha_i(x)$, with $x \in \mathscr{O}_F$ and $i \in \{2, 3, \ldots, n\}$. Prove that

$$I = \mathscr{D}_{F/\mathbb{Q}} \cdot \mathscr{O}_F \,.$$

(b) Prove the following bound

$$\mathrm{disc}_{L/\mathbb{Q}} \supseteq \mathrm{disc}_{F/\mathbb{Q}}^{[L:\mathbb{Q}]/2}$$

of the discriminant ideal of $\mathscr{O}_L$.

2. Let $F, K$ be two number fields. Let $L$ be a number field containing both $F$ and $K$ such that $L$ is the compositum of $F$ and $K$. Prove that

$$\mathscr{D}_{F/\mathbb{Q}} \cdot \mathscr{O}_L \supset \mathscr{D}_{L/K} \,.$$

3. Let $K$ be a number field. Recall that $\Sigma_K$ is the set of all places of $K$.

(a) Show that for every continuous character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$, there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\psi(\mathscr{O}_v) = 1$ for all $v \notin S$.

(b) Show that for every continuous character $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$, there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\chi(\mathscr{O}_v^\times) = 1$ for all $v \notin S$.

(c) Let $F$ be a nonarchimedean locally compact field. Show that every continuous character $\psi : (F, +) \to \mathbb{C}^\times$ is unitary. Does this statement hold for archimedean locally compact fields?

(d) Let $K$ be a global function field. Show that every character of $\mathbb{A}_K \to \mathbb{C}^\times$ is unitary.

(f) Show that for every number field $K$, there exists a character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ which is *not* unitary.

4. Let $K$ be a number field.

(1) Show that there is a natural bijection between (a) the set of all continuous characters of $\mathbb{A}_K$ and (b) the set of all sequences $(\psi_v)_{v \in \Sigma_K}$ indexed by $\Sigma_K$ such that $\psi_v$ is a character of $K_v$ for each $v \in \Sigma_K$ and there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\psi_v(\mathscr{O}_v) = 1$ for every $v \notin S$.

(2) Show that there is a natural bijection between (a) the set of all continuous characters of $\mathbb{A}_K^\times$ and (b) the set of all sequences $(\chi_v)_{v \in \Sigma_K}$ indexed by $\Sigma_K$ such that $\chi_v$ is a character of $K_v^\times$ for each $v \in \Sigma_K$ and there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\chi_v(\mathscr{O}_v^\times) = 1$ for every $v \notin S$.

5. For each prime number $p$, let $\Lambda_p : \mathbb{Q}_p \to \mathbb{Z}[1/p]/\mathbb{Z}$ be the composition of the projection $\mathbb{Q}_p \to \mathbb{Q}_p/Zp$ with the inverse of the isomorphism $\mathbb{Q}_p/Zp \xrightarrow{\sim} \mathbb{Z}[1/p]/\mathbb{Z}$. Let $\Lambda_\infty : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the *negative* of the natural projection. Let $\psi_\mathbb{Q} : \mathbb{A}_\mathbb{Q} \to \mathbb{C}_1^\times$ be the map defined by

$$\psi_\mathbb{Q}(x) = \prod_{v \in \Sigma_\mathbb{Q}} e^{2\pi\sqrt{-1}\cdot\Lambda_v(x_v)} \quad \forall x = (x_v)_{v \in \Sigma_\mathbb{Q}} \in \mathbb{A}_\mathbb{Q}.$$

(a) For every finite extension field $F$ of $\mathbb{Q}_p$, define $\psi_F : F \to \mathbb{C}^\times$ by

$$\psi_F(x) = e^{2\pi\sqrt{-1}\cdot\Lambda_p(\mathrm{Tr}_{F/\mathbb{Q}_p}(x))}.$$

Show $\psi_F$ is a continuous additive unitary character of $F$, and that for every continuous additive character $\psi : F \to \mathbb{C}^\times$, there exists a unique element $y \in F$ such that $\psi(x) = \psi_F(xy)$ for every $x \in F$.

(b) Prove that $\psi_{\mathbb{Q}}$ is continuous and defines a non-trivial unitary character on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$.

(c) For every number field $K$, define a homomorphism $\psi_K : \mathbb{A}_K \to \mathbb{C}_1^\times$ by $\psi_K(x) = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{K/\mathbb{Q}}(x)$. Show that for every $y \in \mathbb{A}_K$, the map $x \mapsto \psi_K(xy)$ is a unitary character of $\mathbb{A}_K$.

(d) Let $K$ be a number field. Show that for every continuous unitary character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ on $\mathbb{A}_K$, there exists a unique element $y \in \mathbb{A}_K$ such that $\psi(x) = \psi_K(xy$ for all $x \in \mathbb{A}_K$.

(e) Show that for every number field $K$ and every continuous character $\phi : \mathbb{A}_K/K \to \mathbb{C}^\times$, there exists a unique element $y \in K$ such that $\psi(x) = \psi_K(xy)$ for all $x \in \mathbb{A}_K$.

6. Find explicitly a Schwartz function $f$ on $\mathbb{A}_{\mathbb{Q},f}$, an idele class character $\chi : \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ and a Haar measure $d^\times x$ on $\mathbb{A}_{\mathbb{A},f}^\times$ such that the associated zeta function

$$\zeta(f, \chi\cdot\omega_s, d^\times x) = \int_{\mathbb{A}_{\mathbb{Q},f}^\times} f(x)\cdot(\chi\omega_s)(x)\,d^\times x$$

is equal to the Dirichlet $L$-function

$$L(s, (\tfrac{-1}{\cdot})) = \sum_{p>2} \left(1 - \left(\tfrac{-1}{p}\right) p^{-s}\right)^{-1}$$

attached to the Legendre symbol $\left(\frac{-1}{\cdot}\right)$.

7. Let $F$ be a number field, $n := [F : \mathbb{Q}]$. Let $a > 0$ be a positive real number. Show that

$$\zeta_F(1+a^{-1}) \le \zeta_{\mathbb{Q}}(1+a^{-1})^n \le (1+a)^n.$$