MATH 620 EXERCISE SET 8, FALL 2016

1. Given an explicit example of an idele class character χ for $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ whose restriction to the archimedian component $\mathbb{R}^{\times}$ of $\mathbb{A}_{\mathbb{Q}}^{\times}$ is equal to the sign character of $\mathbb{R}^{\times}$.

2. Let K be an imaginary quadratic field.

(a) Show that there exists an idele class character $\chi : \mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ such that $\chi_\infty|_{(K\otimes_{\mathbb{Q}}\mathbb{R})^{\times}} \to \mathbb{C}^{\times}$ is equal to the restriction to $(K\otimes_{\mathbb{Q}}\mathbb{R})^{\times}$ of a ring isomorphism $K\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} \mathbb{C}$. In other words, the archimedean component of χ comes from an embedding of K into $\mathbb{C}$.

(b) Suppose that $\chi_1, \chi_2 : \mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ are two idele class characters such that their restrictions to $(K\otimes_{\mathbb{Q}}\mathbb{R})^{\times}$ are equal. Prove that there exists a positive integer n such that $(\chi_1 \cdot \chi_2^{-1})^n$ is trivial.

(c) Give an explicit description of the functional equation for the L-function $L(\chi, s)$, where χ is an idele class character as in (a).

3. Let F be a number field.

(a) Find a Schwartz function f on $\mathbb{A}_F$ such that

$$\zeta_{d^{\times}x}(f, \omega_s) = L_{\mathbb{R}}(s/2)^{r_1} \cdot L_{\mathbb{C}}^{r_2}(s) \cdot \zeta_F(s)$$

Here $d^{\times}x$ is the Haar measure on $\mathbb{A}_F^{\times}$ we specified in lectures: recall that we specify first a Haar measure on dx on $\mathbb{A}_F$, which is a restricted product of Haar measures dx_v's on $(F_v, +)$, normalized by the property that $\int_{\mathbb{A}_F/F} dx = 1$. For each archimedian place v of F, let $d^{\times}x_v$ be $\frac{dx_v}{|x_v|_v}$. For each finite place v of F, let $d^{\times}x_v$ be $(1-q_v^{-1})^{-1} \cdot \frac{dx_v}{|x_v|_v}$. Our preferred multiplicative Haar measure on $\mathbb{A}_F^{\times}$ is the the restricted product of the $d^{\times}x_v$'s.

(b) Deduce from results on local constants that the function $Z_F(s) := |\mathrm{disc}_F|^{s/2} \cdot L_{\mathbb{R}}(s/2)^{r_1} \cdot L_{\mathbb{C}}^{r_2}(s) \cdot \zeta_F(s)$ satisfies the functional equation

$$Z_F(s) = Z_F(1-s).$$

(c) Deduce from the proof of gloal functional equation of zeta functions for $\mathbb{A}_F$ that the Dedekind zeta function $\zeta_F(s)$ has a simple pole at $s-1$ with residue

$$\kappa_F = \frac{2^{r_1}(2\pi)^{r_2} R_F}{|\mu(F)| \cdot |\mathrm{disc}_F|^{1/2}}$$

(d) Determine the order of zero/pole of $\zeta_F(s)$ at non-positive integers.

(e) Let χ be an idele class character of finite order for $\mathbb{A}_F^{\times}/F^{\times}$. Give an explicit form of the functional equation for $L_F(\chi, s) = L_F(\chi\,\omega_s)$.

4. (This problem leads you through Hecke's proof of the funcitonal equation for Dedekind zeta functions.) The function f you obtained in 1(a) above is likely to be a product of a function f_∞ on $F_\infty := F\otimes_{\mathbb{Q}}\mathbb{R}$ and the characteristic function of $\mathscr{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Assume this is the case.

(a) Let $H(F)$ be the class group of $\mathscr{O}_F$. The canonical surjective homomorphism $\mathbb{A}_K^\times \to H(F)$ gives a decomposition of $\mathbb{A}_F^\times$ into a disjoint union of cosets for the open subgroup $F \cdot (F_\infty^\times \times (\mathscr{O}_F \times_\mathbb{Z} \hat{\mathbb{Z}})^\times$. Correspondingly the integral $\int_{\mathbb{A}_F^\times} f(x)\, \omega_s(x)\, d^\times x$ becomes a sum of h_F integrals, indexed by the class group $H(F)$. For each of the h_F integrals, first integrate (i.e. sum) over $F^\times$. You should see a family of $[F:\mathbb{Q}]$-variable theta series appearing. (In the sense that the sum over $F^\times$ is the theta series minus the constant term of the theta series.)

Note: By a theta series we mean a series of the form $\sum_{\xi\in\Gamma} e^{-Q(\xi)}$, where V is a finite dimensional vector space over $\mathbb{R}$, Γ is a co-compact discrete subgroup of V, and Q is an $\mathbb{R}$-valued positive definite quadratic form on V.

(b) Rewrite each of the h_F integrals as an integral over $\prod_{v\in\Sigma_{F,\infty}} \mathbb{R}_{>0}^\times \cong (\mathbb{R}_{>0}^\times)^{r_1+r_2}$, where the integrand involves a theta series and a suitable power of the norm function on $(\mathbb{R}_{>0}^\times)^{r_1+r_2}$.

(c) Apply the Poisson summation formula, which results in a functional equation relating the theta series for a quadratic form Q to a theta series for a quadratic form Q' on $V' := \mathrm{Hom}_\mathbb{R}(V,\mathbb{R})$ and the lattice $\Gamma' := \Gamma^\perp$ in V'. The original integral now becomes an integral whose integrand has three terms, one term is a theta series without the constant term, plus two other explicit terms. The last two terms are easily integrated (absolutely convergent for $\mathrm{Re}(s) > 1$.

(d) Use Fubini theorem for the integral left. Integrate first over the norm-one subgroup of $(\mathbb{R}_{>0}^\times)^{r_1+r_2}$, then over the quotient by the norm-one subgroup. Note that the quotient is naturally isomorphic to $\mathbb{R}_{>0}^\times$). In $\int_0^\infty = \int_0^1 + \int_1^\infty$, make a change of variable $t \mapsto 1/t$. Congratulations! You have completed Hecke's proof of the functional equation. At this point you should see that you have actually proved h_F functional equations. Each of these functional equations relates a partial zeta function $\zeta_\mathfrak{a}(s)$, which is a sum over ideals in a fixed element $\mathfrak{a}$ in the ideal class group $H(F)$, to $\zeta_{\mathfrak{a}^{-1}}(1-s)$.