Exercise 2

1. Work over a field \( k \). Let \( T \subset \mathbb{P}^2 \) be the “triangle” defined by \( x_0x_1x_2 = 0 \), a closed subscheme. Let \( f : \mathbb{P}^2 - T \rightarrow \mathbb{P}^2 - Z \) be the isomorphism defined in projective coordinates by

\[
(x_0 : x_1 : x_2) \mapsto (\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}).
\]

Let \( Z \) be the Zariski closure of the graph of \( f \) in \( \mathbb{P}^2 \times \text{Spec}(k) \mathbb{P}^2 \), a closed subscheme of \( \mathbb{P}^2 \times \text{Spec}(k) \mathbb{P}^2 \). Let \( pr_1 : Z \rightarrow \mathbb{P}^2 \) be the projection to the first factor of \( \mathbb{P}^2 \times \text{Spec}(k) \mathbb{P}^2 \), thought of as the source of the birational map \( f \). Relate \( pr_1 : Z \rightarrow \mathbb{P}^2 \) to a suitable blowing up of \( \mathbb{P}^2 \).

2. Give an example of a scheme \( X \) with two affine open subsets \( U \) and \( V \) such that \( U \cap V \) is not affine.

3. Work over a base field \( k \). Let \( X \) be a smooth quadric in \( \mathbb{P}^3 \), \( x_0 \) be a \( k \)-rational point of \( X \), and let \( X \longrightarrow \mathbb{P}^2 \) be the projection from \( x_0 \) to a plane disjoint from \( x_0 \), a rational map which is regular on \( X \setminus \{ x_0 \} \).

   (i) Show that \( g \) does not extend to a morphism on \( X \).
   (ii) Show that \( g \) is a birational map.
   (iii) Determine all \( \mathbb{P}^1 \)'s contracted by \( g \).
   (iv) Let \( \alpha : B \rightarrow X \) be the blowing up of \( X \) at \( x_0 \). Show that the birational map \( g \) induces a morphism \( \beta : B \rightarrow \mathbb{P}^2 \).
   (v) Let \( y_1 \) and \( y_2 \) be the image in \( \mathbb{P}^2 \) of the two lines in \( X \) contracted under \( g \). Show that \( B \) is isomorphic to the blowing up of \( \mathbb{P}^2 \) at \( y_1 \) and \( y_2 \).
   (vi) Show that the birational map \( \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \) is given by the linear system of conics on \( \mathbb{P}^2 \) passing through \( y_1 \) and \( y_2 \).
   [Note: We have not defined the notion of a linear system so far. So you can leave this part aside and come back when you know what a linear system is.]
   (vii) Show that \( X \) is not isomorphic to the blowing up of \( \mathbb{P}^2 \) centered at a closed point.

4. Let \( k \) be a field, \( V \) be a finite dimensional vector space over \( k \). Let \( \rho : \text{GL}_n \rightarrow \text{GL}(V) \) be a \( k \)-linear rational representation of \( \text{GL}_n \) on \( V \), i.e. the homomorphism \( \rho \) is a \( k \)-morphism of group schemes over \( k \). Suppose that \( v \in V \) is a vector fixed by the subgroup \( B \) of all upper-triangular elements in \( \text{GL}_n \). Prove that \( v \) is fixed by \( \text{GL}_n \).
   [Hint: The quotient variety \( \text{GL}_n / B \) is proper over \( k \).]

5. Work over a field \( k \). Let \( H \) be a hyperplane in \( \mathbb{P}^n \), \( n \geq 2 \). Let \( Z \subset H \) be a smooth hypersurface in \( H \) of degree \( d \), \( n \geq 2 \). Let \( f : X \rightarrow \mathbb{P}^n \) be the blowing of \( \mathbb{P}^n \) with center \( Z \). Let \( Y \) be the strict transform of \( H \), i.e. \( Y \) is the closure in \( X \) of \( f^{-1}(H - Z) \), where \( H - Z \) denotes the complement of \( Z \) in \( H \). By the universal property of blowing ups, the \( \mathcal{O}_X \)-module \( \mathcal{I} := f^{-1} \mathcal{I}_Z \cdot \mathcal{O}_X \), or the ideal in \( \mathcal{O}_X \) generated by the image of the sheaf of ideals \( \mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^n} \) for \( Z \subset \mathbb{P}^n \), is an invertible \( \mathcal{O}_X \)-module isomorphic to the sheaf “\( \mathcal{O}_X(1) \)” on \( X = \mathcal{O}_X \mathcal{O}_Y \oplus_{n \geq 0} \mathcal{I}^n \). Show that \( Y \) is isomorphic to \( H \) under the morphism \( f \), and \( \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \) is isomorphic to \( f^* \mathcal{O}_{\mathbb{P}^n}(-d) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \).
6. Let $X$ be a noetherian integral scheme, $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and let $f \in \Gamma(X, \mathcal{L}^\otimes n)$ be a global section of $\mathcal{L}^\otimes n$, $n \geq 2$. Let $B \subset X$ be the Cartier divisor defined by $f$, so that $f$ defines an isomorphism $\mathcal{L}^\otimes n \cong \mathcal{O}_X(B)$. Let $L := \text{spec} \left( \bigoplus_{m \geq 0} \mathcal{L}^\otimes (-m) \right) \xrightarrow{\pi} X$, thought of as the total space of the line bundle over $X$ whose local sections is $\mathcal{L}$. Denote by $T$ the tautological global section of $\tilde{\pi}^* \mathcal{L}$, corresponding to the canonical element

$$1 \in \Gamma(X, \mathcal{L}^\otimes (-1) \otimes \mathcal{L}) \subset \bigoplus_{m \geq 0} \Gamma(X, \mathcal{L}^\otimes (-m) \otimes \mathcal{L}) = \Gamma(L, \tilde{\pi}^* \mathcal{L}).$$

The cyclic cover of order $n$ of $X$ attached to the triple $(X, \mathcal{L}, f)$ is by definition the divisor $Y \subset L$ of the section $T^n - \pi^* f \in \Gamma(L, \pi^* \mathcal{L}^\otimes n)$. Let $\pi : Y \rightarrow X$ be the finite locally free morphism induced by $\tilde{\pi}$. Let $B_1 \subset Y$ be the Cartier divisor in $Y$ attached to the $T | _Y \in \Gamma(Y, \pi^* \mathcal{L})$, the image in in $\Gamma(Y, \pi^* \mathcal{L})$ of the tautological section of $\tilde{\pi}^* \mathcal{L}$.

(i) Show that $\pi_\ast \mathcal{O}_Y$ is isomorphic to $\bigoplus_{0 \leq m \leq n-1} \mathcal{L}^\otimes (-m)$ as an $\mathcal{O}_X$-module.

(ii) Verify that $B_1$ is the inverse image of $B$ in $Y$, and we have a natural isomorphism $\pi^* \mathcal{L} \cong \mathcal{O}_Y(B_1)$. Consequently $\pi^* \mathcal{O}_X(B) \cong \mathcal{O}_Y(B_1)^\otimes n$.

7. Work over a field $k$ of characteristic $\neq 2$. Let $B \subset \mathbb{P}^2$ be a smooth conic curve defined by a homogeneous quadratic polynomial $f(x,y,z)$. Let $\pi : Y \rightarrow \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ attached to the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), f)$, a smooth projective surface.

(i) If $l$ is a line in $\mathbb{P}^2$ meeting $B$ at two distinct points, then $\pi^{-1}(l)$ is a smooth curve in $Y$ and $\deg(\mathcal{L} | _{\pi^{-1}(l)}) = 2$.

(ii) If $l$ is a tangent line to $B$, then $\pi^{-1}(l)$ is the union $\tilde{t}_1 \cup \tilde{t}_2$ of two smooth curves in $Y$ meeting transversally at a point. Moreover $\deg(\mathcal{L} | _l) = 1$ for $l = 1, 2$.

(iii) Show that $B$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

8. Let $X = F(a_1, \ldots, a_n) := \rho \text{proj}_{\mathbb{P}^1}(S^\ast (\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathbb{P}^1(a_n)))$. Assume for simplicity that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $\pi : X \rightarrow \mathbb{P}^1$ be the structural morphism, so that $X$ is a family of $\mathbb{P}^{n-1}$’s parametrized by $\mathbb{P}^1$. Denote by $\mathcal{O}_X(1)$ the universal invertible quotient $\mathcal{O}_X$-module of

$$\pi^\ast (\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathbb{P}^1(a_n)).$$

For every local ring $(R, m)$, let $S_R$ be the set

$$\{ (t_0, t_1 : x_2 : \ldots : x_n) \in R^{n+2} \mid t_0 R + t_1 R = R, x_1 R + \cdots + x_n R = R \}$$

modulo the equivalence relation generated by

$$(t_0, t_1 : x_2 : \ldots : x_n) \sim (t_0, t_1 : \mu x_1 : x_2 : \ldots : \mu x_n) \quad \mu \in R^\times$$

$$(t_0, t_1 : x_2 : \ldots : x_n) \sim (\lambda t_0, \lambda t_1 : \lambda^{-a_1} x_1 : \lambda^{-a_2} x_2 : \ldots : \lambda^{-a_n} x_n) \quad \lambda \in R^\times$$

(i) Show that there is a functorial bijection, between $X(R)$ and the set $S_R$ for every local ring $(R, m)$.

(ii) Show by a counterexample that the above description of $X(R)$ is no longer valid when $R$ is not a local ring.